Symmetries and supersymmetries of the Dirac operators in curved spacetimes

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Abstract

It is shown that the main geometrical objects involved in all the symmetries or supersymmetries of the Dirac operators in curved manifolds of arbitrary dimensions are the Killing vectors and the Killing-Yano tensors of any ranks. The general theory of external symmetry transformations associated to the usual isometries is presented, pointing out that these leave the standard Dirac equation invariant providing the correct spin parts of the group generators. Furthermore, one analyses the new type of symmetries generated by the covariantly constant Killing-Yano tensors that realize certain square roots of the metric tensor. Such a Killing-Yano tensor produces simultaneously a Dirac-type operator and the generator of a one-parameter Lie group connecting this operator with the standard Dirac one. In this way the Dirac operators are related among themselves through continuous transformations associated to specific discrete ones. It is shown that the groups of this continuous symmetry can be only $U(1)$ or $SU(2)$, as those of the (hyper-)Kähler spaces, but arising even in cases when the requirements for these special geometries are not fulfilled. Arguments are given that for the non-Kählerian manifolds it is convenient to enlarge this $SU(2)$ symmetry up to a $SL(2,\mathbb{C})$ one through complexification. In other respects, it is pointed out that the Dirac-type operators can form $\mathcal{N}=4$ superalgebras whose automorphisms combine external symmetry transformations with those of the mentioned $SU(2)$ or $SL(2,\mathbb{C})$ groups. The discrete symmetries are also studied obtaining the discrete groups $\mathbb{Z}_4$ and $\mathbb{Q}$. To exemplify, the Euclidean Taub-NUT space with its Dirac-type operators is presented in much details. Finally the properties of the Dirac-type operators of the Minkowski spacetime are briefly discussed in the Appendix B.

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1 Introduction

The quantum physics in curved backgrounds needs to use algebras of operators acting on spaces of vector, tensor or spinor fields whose properties depend on the geometry of the manifolds where these objects are defined. A crucial problem is to find the symmetries having geometrical sources able to produce conserved quantities related to operators with specific algebraic properties. The problem is not trivial since, beside the evident geometrical symmetry given by isometries, there are different types of hidden symmetries frequently associated with supersymmetries that deserve to be carefully studied.

The isometries are related to the existence of the Killing vectors that give rise to the orbital operators of the scalar quantum theory commuting with that of the free field equation. In the theories with spin these operators get specific spin terms whose form is strongly dependent on the local non-holonomic frames we choose by fixing the gauge. For the Dirac field, these spin parts are known in any chart and arbitrary tetrad gauge fixing of the four-dimensional manifolds [1, 2]. However, with these results one cannot say that this problem is generally solved, for fields with any spin obeying different free or coupled field equations. For this reason the theory of isometries was extended to that of the external symmetry which allows one to pick up well-defined conserved quantities in theories with matter fields of *any spin* [3, 4]. Here we extend the theory of external symmetries to the Dirac theory in manifolds of any dimensions.

Another type of geometrical objects related to the so called hidden symmetries or several specific supersymmetries are the Killing-Yano (K-Y) tensors [5] and the Stäckel-Killing (S-K) tensors of any rank. The K-Y tensors play an important role in theories with spin and especially in the Dirac theory on curved spacetimes where they produce first order differential operators, called Dirac-type operators, which anticommute with the standard Dirac one, D [1, 6]. Another virtue of the K-Y tensors is that they enter as square roots in the structure of several second rank S-K tensors that generate conserved quantities in classical mechanics or conserved operators which commute with D. The construction of Ref. [1] depends upon the remarkable fact that the S-K tensors must have square root in terms of K-Y tensors in order to eliminate the quantum anomaly and produce operators commuting with D [7]. These attributes of the K-Y tensors lead to an efficient mechanism of supersymmetry especially when the S-K tensor is proportional with the metric tensor and the corresponding roots are covariantly constant K-Y tensors. Then each tensor of this type, f^i, gives rise to a Dirac-type operator, D^i, representing a supercharge of a non-trivial superalgebra $\{D^i, D^j\} \propto D^2\delta_{ij}$ [8]. It was shown that D^i can be produced by covariantly constant K-Y tensors having not only real-valued components but also complex ones [9, 10]. This represents an extension of the Kählerian manifolds that seems to be productive for the Dirac theory since it permits to construct superalgebras of Dirac-type operators even in the Minkowski spacetime which is not Kählerian, having only complex-valued covariantly constant K-Y tensors [10, 11]. For this reason, in what follows we shall consider such more general tensors, called *unit roots* (instead of complex structures) since all of them are roots of the metric tensor. We note that the complex structures defining Kählerian geometries are special automorphisms of the tangent fiber bundle while the unit roots we consider here are automorphisms of the *complexified* tangent fiber bundle. A part of this paper is devoted to the theory of the Dirac-type operators generated by unit roots [11, 12].

It is known that in four-dimensional manifolds the standard Dirac operator and the Dirac-type ones can be related among themselves through continuous or discrete transformations [12, 10]. It is interesting that there are only two possibilities, namely either transformations of the $U(1)$ group associated with the discrete group $\mathbb{Z}_4$ or $SU(2)$ transformations and discrete ones of the quaternionic group $\mathbb{Q}$ [12, 10, 11]. Particularly, in the case when the roots are real-valued (complex structures) the first type of symmetry is proper to Kähler manifolds while the second largest one is characteristic for hyper-Kähler geometries [12]. The problem is what happens in the case of manifolds with larger number of dimensions allowing complex-valued roots. We have shown that, in general, there are no larger symmetries of this type [11] but here we point out how these could be embedded with the isometries. Other new result is that in non-Kählerian manifolds with complex-valued unit roots this operation requires to enlarge the $SU(2)$ symmetry up to a $SL(2,\mathbb{C})$ one using complexification.

The typical example is the Euclidean Taub-NUT space which is a hyper-Kähler manifold possessing three covariantly constant K-Y tensors with real-valued components which constitute a hypercomplex structure generating a $\mathcal{N}=4$ superalgebra of Dirac-type operators [8], in a similar way as in semi-classical spinning models [13, 14]. Moreover, each involved K-Y tensor is a unit root of the metric tensor as it results from the definition of the Kählerian geometries (given in Appendix A). It is worth pointing out that the Euclidean Taub-NUT space has, in addition, a non-covariantly constant K-Y tensor related to its specific hidden symmetry showed off by the existence of a conserved Runge-Lenz operator that can be constructed with the help of the Dirac-type operators produced by the four K-Y tensors of this space [15, 16]. In Euclidean Taub-NUT space there are no quantum anomalies [17] and one obtains a rich algebra of conserved observables [18] that offers many possibilities to choose sets of commuting operators defining quantum modes [19, 8]. On the other hand, hereby one can select or build superalgebras, dynamical algebras typical for the Keplerian problems [16], or even interesting infinite loop algebras. Our last objective here is to present the complete Dirac theory in the Taub-NUT background including our new results in this domain.

The paper is organized as follows. We start in the second section with the construction of a simple version of the Dirac theory in manifolds of any dimensions, introducing the group of the external symmetry in non-holonomic local frames as the universal covering group of the isometry one [3]. In this way we can define the spinor representation recovering the specific form of its generators Ref. [1] in a suitable context that allows us to use the Noether theorem for deriving conserved quantities [4]. In the next section we present the theory of the Dirac-type operators produced by the unit roots. The continuous and discrete symmetries of these operators are studied showing that there exists either an $U(1)$ symmetry associated to a single unit root or a $SU(2)$ one of of the Dirac-type operators produced by triplets of unit roots. Moreover, we point out that the these triplets give rise to triplets of Dirac-type operators, D^i, $i = 1, 2, 3$ anticommuting with D and among themselves too, forming thus a basis of a $\mathcal{N} = 4$ superalgebra. Furthermore, we show that in the case of the hyper-Kähler manifolds, the automorphisms of these superalgebras combine the mentioned $SU(2)$ specific transformations with those of a representation of the group of external symmetry induced by $SO(3)$ rotations among the triplet elements. For the superalgebras generated in non-Kählerian manifolds the problem is more complicated since there the isometry transformations are induced by the complexified group $O(3)_c$ which forces us to consider complexified groups and Lie algebras. The discrete symmetries associated with the continuous ones, $\mathbb{Z}_4$ and $\mathbb{Q}$, as well as the parity and the charge conjugation are also presented. The last section is devoted to the theory of the Dirac operators in the Euclidean Taub-NUT space. Adopting a group theoretical point of view, we start with the integral form of the isometry transformations we have recently obtained and the orbital angular momentum operator that deals with them [20]. Moreover, we show why in the usual gauge the whole theory presents a $SO(3)$ global symmetry and we review the principal operators of the scalar Klein-Gordon, Pauli and Dirac theory. Some important algebraic features are pointed out giving a special attention to an association among Pauli and Dirac conserved operators [18] that simplifies the algebraic calculus leading to possible infinite loop superalgebras. Different discrete quantum Dirac modes are constructed with the help of reducible representations of the dynamical $so(4)$ algebra [16] and using new types of spherical harmonics and spinors [19, 8]. Finally, in two appendices we briefly present the Kählerian manifolds and the properties of the Dirac-type operators of the Minkowski spacetime.

2 The external symmetry of the Dirac field

The relativistic covariance in the sense of general relativity is too general to play the same role as the Lorentz or Poincaré covariance in special relativity [21]. In other respects, the gauge covariance of the theories with spin represents another kind of general symmetry that is not able to produce itself conserved observable [22]. Therefore, if we look for sources of symmetries able to generate conserved quantities, we have to concentrate first on isometries that point out the spacetime symmetry giving us the specific Killing vectors [22, 23, 24]. The physical fields minimally coupled with the gravitational one take over this symmetry, transforming according to different representations of the isometry group. In the case of the scalar vector or tensor fields these representations are completely defined by the well-known rules of the general coordinate transformations since the isometries are in fact particular coordinate transformations. However, the behavior under isometries of the fields with half integer spin is more complicated since their transformation rules explicitly depend on the gauge fixing. The specific theory of this type of transformations is the recent theory of external symmetry we present in this section [3].

2.1 Clifford algebra and the gauge group

The theory of the Dirac spinors in arbitrary dimensions depends on the choice of the manifold and Clifford algebra. Bearing in mind that the irreducible representations of the Clifford algebra can have only an odd number of dimensions, we consider a $2l+1$-dimensional pseudo-Riemannian manifold M_{2l+1} whose flat metric $\tilde{\eta}$ (of its pseudo-Euclidean model) has the signature (m_+, m_-) where $m_+ + m_- = m = 2l+1$. This is the *maximal* manifold that can be associated to the $2l+1$-dimensional Clifford algebra [25] acting on the 2^l-dimensional space Ψ of the complex spinors $\psi = \tilde{\varphi}_1 \otimes \tilde{\varphi}_2 ... \otimes \tilde{\varphi}_l$ built using complex two-dimensional Pauli spinors $\tilde{\varphi}$. In this algebra we start with the standard Euclidean basis formed by the hermitian matrices $\tilde{\gamma}^A = (\tilde{\gamma}^A)^+$ $(A, B, ... = 1, 2, ..., m)$ that obey $\{\tilde{\gamma}^A, \tilde{\gamma}^B\} = 2\delta^{AB}\mathbf{1}$ where $\mathbf{1}$ is the identity matrix. Furthermore, we define the suitable basis corresponding to the metric $\tilde{\eta}$ as

$$\gamma^A = \left\{ \begin{array}{lll} \tilde{\gamma}^A & \text{for} & A = 1, 2, ..., m_+ \\ i\tilde{\gamma}^A & \text{for} & A = m_+ + 1, m_+ + 2, ..., m \end{array} \right. , \tag{1}$$

such that

$$\{\gamma^A, \gamma^B\} = 2\tilde{\eta}^{AB}\mathbf{1} \,. \tag{2}$$

Since the first m_+ matrices γ^A remain hermitian while the m_- last ones become anti-hermitian, it seems that the unitaryness of the theory is broken. However, this can be restored replacing the usual Hermitian adjoint with the generalized Dirac adjoint [26].

Definition 1 *We say that $\overline{\psi} = \psi^+\gamma$ is the generalized Dirac adjoint of the field ψ if the hermitian matrix $\gamma = \gamma^+$ satisfies the condition $(\gamma)^2 = \mathbf{1}$ and all the matrices γ^A are either self-adjoint or anti self-adjoint with respect to this operation, i.e. $\overline{\gamma}^A = \gamma(\gamma^A)^+\gamma = \pm\gamma^A$.*

It is clear that the matrix γ play here the role of *metric operator* giving the generalized Dirac adjoint of any square matrix X as $\overline{X} = \gamma X^+ \gamma$.

Theorem 1 *The metric operator can be represented as the product* $\gamma=\epsilon\,\gamma^1\gamma^2...\gamma^{m_+}$ *with the phase factor*

$$\epsilon=\left\{\begin{array}{lll}(i)^{\frac{m_+-1}{2}} & \text{for odd} & m_+<m\\ (i)^{\frac{m_+}{2}} & \text{for even} & m_+<m\end{array}\right. . \tag{3}$$

Proof: In the special case of the Euclidean metric (when $m_-=0$) we have the trivial solution $\gamma=\mathbf{1}$. Otherwise, the algebraic properties of the matrix γ depends on m_+ such that for m_+ taking odd values we have the following superalgebra

$$\begin{array}{rl}[\gamma,\,\gamma^A]=0 & \text{for}\quad A=1,2,...,m_+\,,\\ \{\gamma,\,\gamma^A\}=0 & \text{for}\quad A=m_++1,...,m\,,\end{array} \tag{4}$$

while for even m_+ the situation is reversed. Consequently, one can verify that

$$\overline{\gamma^A}=\left\{\begin{array}{lll}\gamma^A & \text{for odd} & m_+\\ -\gamma^A & \text{for even} & m_+\end{array}\right. ,\quad A=1,2,...,m\,, \tag{5}$$

which means that from the point of view of the Dirac adjoint all the matrices γ^A have the same behavior, being either self-adjoint or anti self-adjoint. Thus the unitaryness of the theory is guaranteed. ∎
In what follows we consider that m_+ is odd representing the number of time-like coordinates and match all the phase factors according to self-adjoint gamma-matrices.

Remark 1 *When these matrices are anti self-adjoint (because of an even* m_+*) it suffices to change* $\gamma^A\to\pm i\gamma^A$ *in the formulas of all the operators one defines. The same procedure is indicated when one works with self-adjoint gamma-matrices but metrics of reversed signature, with* m_+ *space-like coordinates.*

The isometry group $G(\tilde\eta)=O(m_+,m_-)$ of the metric $\tilde\eta$, with the mentioned signature, is the *gauge* group of the theory defining the principal fiber bundle. This is a pseudo-orthogonal group that admits an universal covering group $\boldsymbol{G}(\tilde\eta)$ which is simply connected and has the same Lie algebra we denote by $\boldsymbol{g}(\tilde\eta)$. The group $\boldsymbol{G}(\tilde\eta)$ is the model of the spinor fiber bundle that completes the spin structure we need. In order to avoid complications due to the presence of these two groups we consider here that the basic piece is the group $\boldsymbol{G}(\tilde\eta)$, denoting by $[\omega]$ their elements in the standard *covariant* parametrization given by the skew-symmetric real parameters $\omega_{AB}=-\omega_{BA}$. Then the identity element of $\boldsymbol{G}(\tilde\eta)$ is $1=[0]$ and the inverse of $[\omega]$ with respect to the group multiplication reads $[\omega]^{-1}=[-\omega]$.

Definition 2 *We say that the gauge group is the* vector *representation of* $\boldsymbol{G}(\tilde\eta)$ *and denote* $G(\tilde\eta)=vect[\boldsymbol{G}(\tilde\eta)]$*. The representation* $spin[\boldsymbol{G}(\tilde\eta)]$ *carried by the space* Ψ *and generated by the spin operators*

$$S^{AB}=\frac{i}{4}\left[\gamma^A,\,\gamma^B\right] \tag{6}$$

is called the spinor *representation of* $\boldsymbol{G}(\tilde{\eta})$*. The spin operators are the basis generators of the spinor representation* $spin[\boldsymbol{g}(\tilde{\eta})]$ *of the Lie algebra* $\boldsymbol{g}(\tilde{\eta})$*.*

In general, the spinor representation is reducible. Its generators are self-adjoint, $\overline{S}^{AB} = S^{AB}$, and satisfy

$$
\begin{aligned}
[S^{AB}, \gamma^C] &= i(\tilde{\eta}^{BC}\gamma^A - \tilde{\eta}^{AC}\gamma^B)\,, & (7)\\
[S_{AB}, S_{CD}] &= i(\tilde{\eta}_{AD}\, S_{BC} - \tilde{\eta}_{AC}\, S_{BD} + \tilde{\eta}_{BC}\, S_{AD} - \tilde{\eta}_{BD}\, S_{AC})\,, & (8)
\end{aligned}
$$

as it results from Eqs. (2) and (6). It is obvious that Eq. (8) gives just the canonical commutation rules of a Lie algebra isomorphic with that of the groups $G(\tilde{\eta})$ or $\boldsymbol{G}(\tilde{\eta})$. The spinor and vector representations are related between themselves through the following

Theorem 2 *For any real or complex valued skew-symmetric tensor* $\omega_{AB} = -\omega_{BA}$ *the matrix*

$$T(\omega) = e^{-iS(\omega)}\,, \quad S(\omega) = \frac{1}{2}\omega_{AB}S^{AB}\,, \tag{9}$$

transforms the gamma-matrices according to the rule

$$[T(\omega)]^{-1}\gamma^A T(\omega) = \Lambda^{A\cdot}_{\cdot B}(\omega)\gamma^B\,, \tag{10}$$

where

$$\Lambda^{A\cdot}_{\cdot B}(\omega) = \delta^A_B + \omega^{A\cdot}_{\cdot B} + \frac{1}{2}\,\omega^{A\cdot}_{\cdot C}\omega^{C\cdot}_{\cdot B} + ... + \frac{1}{n!}\,\underbrace{\omega^{A\cdot}_{\cdot C}\omega^{C\cdot}_{\cdot C'}...\omega^{D\cdot}_{\cdot B}}_{n} + ...\,. \tag{11}$$

Proof: All these results can be obtained using Eqs. (2) and (6). ∎
The real components ω_{AB} are the parameters of the covariant basis of the Lie algebra $\boldsymbol{g}(\tilde{\eta})$ giving all the transformation matrices $T(\omega) \in spin[\boldsymbol{G}(\tilde{\eta})]$ and $\Lambda(\omega) \in vect[\boldsymbol{G}(\tilde{\eta})]$. Hereby we see that the spinor representation $spin[\boldsymbol{G}(\tilde{\eta})]$ is unitary since for $\omega \in \mathbb{R}$ the generators $S(\omega) \in spin[\boldsymbol{g}(\tilde{\eta})]$ are self-adjoint, $\overline{S}(\omega) = S(\omega)$, and the matrices $T(\omega)$ are unitary with respect to the Dirac adjoint satisfying $\overline{T}(\omega) = [T(\omega)]^{-1}$.

The covariant parameters ω can also take complex values. Then this parametrisation spans the *complexified* group of $\boldsymbol{G}(\tilde{\eta})$, denoted by $\boldsymbol{G}_c(\tilde{\eta})$, and the corresponding vector and (non-unitary) spinor representations. Obviously, in this case the Lie algebra is the complexified algebra $\boldsymbol{g}_c(\tilde{\eta})$. We note that from the mathematical point of view $G(\tilde{\eta}) = vect[\boldsymbol{G}(\tilde{\eta})]$ is the group of automorphisms of the tangent fiber bundle $\mathcal{T}(M_m)$ of M_m while the transformations of $G_c(\tilde{\eta}) = vect[\boldsymbol{G}_c(\tilde{\eta})]$ are automorphisms of the complexified tangent fiber bundle $\mathcal{T}(M_m) \otimes \mathbb{C}$.

2.2 The Dirac theory

With these preparations, the gauge-covariant theory of the Dirac field can be formulated on any submanifold of $M_{m=2l+1}$, like, for example, in the usual (1+3)-dimensional spacetimes immersed in the five-dimensional manifold of the Kaluza-Klein theory (with $l = 2$). We consider the general case of the Dirac theory on any submanifold $M_n \subset M_m$ of dimension $n \le m$ whose flat metric η is a part (or restriction) of the metric $\tilde{\eta}$, having the signature (n_+, n_-), with $n_+ \le m_+$, $n_- \le m_-$ and $n_+ + n_- = n$, such that the gauge group is $G(\eta) = vect[\boldsymbol{G}(\eta)] = O(n_+, n_-)$. In M_n we choose a local chart (i.e. natural frame) with coordinates x^μ, $\alpha, ..., \mu, \nu, ... = 1, 2, ..., n$, and introduce local orthogonal non-holonomic frames using the gauge fields (or "vilbeins") $e(x)$ and $\hat{e}(x)$, whose components are labeled by local (hated) indices, $\hat{\alpha}, ...\hat{\mu}, \hat{\nu}, ... = 1, 2, ..., n$, that represent a subset of the Latin capital ones, eventually renumbered. The local indices have to be raised or lowered by the metric η. The fields e and $\hat{e}$ accomplish the conditions

$$e^\mu_{\hat{\alpha}} \hat{e}^{\hat{\alpha}}_\nu = \delta^\nu_\mu \,, \quad e^\mu_{\hat{\alpha}} \hat{e}^{\hat{\beta}}_\mu = \delta^{\hat{\beta}}_{\hat{\alpha}} \tag{12}$$

and orthogonality relations as $g_{\mu\nu} e^\mu_{\hat{\alpha}} e^\nu_{\hat{\beta}} = \eta_{\hat{\alpha}\hat{\beta}}$. With their help the metric tensor of M_n can be put in the form $g_{\mu\nu}(x) = \eta_{\hat{\alpha}\hat{\beta}} \hat{e}^{\hat{\alpha}}_\mu(x) \hat{e}^{\hat{\beta}}_\nu(x)$.

Definition 3 *We call physical spacetimes the manifolds $M_n = M_{d+1}$ having only one time-like coordinate $x^0 = t$ and $d = n - 1$ space-like ones, $\boldsymbol{x} = (x^1, x^2, ...x^d)$, with metrics of the signature $(1, d)$. In addition, we assume that these manifolds are orientable and time-orientable [23].*

The next step is to choose a suitable representation of the n matrices $\gamma^{\hat{\alpha}}$ obeying Eq. (2) and to calculate the spin matrices $S^{\hat{\alpha}\hat{\beta}}$ defined by Eq. (6). Now these are the basis generators of the spinor representation $spin[\boldsymbol{g}(\eta)]$ of the Lie algebra $\boldsymbol{g}(\eta)$, corresponding to the metric η. If $n < m$ there are many matrices, $\gamma^{n+1}, ..., \gamma^m$, which anticommutes with all the n matrices $\gamma^{\hat{\alpha}}$ one uses for the Dirac theory in M_n. We can select one of these extra gamma-matrices denoting it by γ^{ch} and matching its phase factor such that $(\gamma^{ch})^2 = \mathbf{1}$ and $(\gamma^{ch})^+ = \gamma^{ch}$. This matrix obeying

$$\left\{\gamma^{ch}, \, \gamma^{\hat{\mu}}\right\} = 0 \,, \quad \hat{\mu} = 1, 2, ..., n \,, \tag{13}$$

is called the *chiral* matrix since it plays the same role as the matrix γ^5 in the usual Dirac theory, helping us to distinguish between even and odd matrices or matrix operators.

Definition 4 *One says that a matrix operator acting on Ψ is* even *whenever it commutes with γ^{ch} and is* odd *if it anticommutes with this matrix.*

The matrix γ can be either even (for even m_+) or odd (when m_+ is odd). These two different situations lead to self-adjoint or anti self-adjoint chiral matrices, such that it is convenient to use the chiral phase factor $\epsilon_{ch} = \pm 1$ giving $\overline{\gamma}^{ch} = \epsilon_{ch} \gamma^{ch}$. In general, when $m > n$ we can define one or even many chiral matrices different from γ but for $n = m = 2l + 1$ we must take $\gamma^{ch} = \gamma$. In any case, the space of the Dirac spinors can be split in its left and right-handed parts, $\Psi = \Psi_L \oplus \Psi_R$ ($\Psi_L = P_L \Psi$, $\Psi_R = P_R \Psi$), using the traditional projection operators

$$P_L = \frac{1}{2}\left(\mathbf{1} - \gamma^{ch}\right) \,, \quad P_R = \frac{1}{2}\left(\mathbf{1} + \gamma^{ch}\right) \,. \tag{14}$$

In general, any matrix operator $X : \Psi \to \Psi$ can be written as the sum $X = X_{even} + X_{odd}$, between its even part $X_{even} = P_L X P_L + P_R X P_R$ and the odd one $X_{odd} = P_L X P_R + P_R X P_L$. When one intends to exploit this mechanism it is convenient to use the *chiral* representation of the gamma-matrices where

$$\gamma^{ch} = \left(\begin{array}{cc} -\mathbf{1}_L & 0 \\ 0 & \mathbf{1}_R \end{array} \right) \tag{15}$$

is diagonal and all the matrices $\gamma^{\hat\mu}$ are off-diagonal. The notations $\mathbf{1}_L$ and $\mathbf{1}_R$ stand for the identity matrices on the spaces of spinors Ψ_L and Ψ_R respectively. The gamma-matrices and the metric operator γ in the chiral representation must be calculated in each concrete case separately since they depend on the metric signature. In what follows we assume that in the physical spacetimes defined by Definition 3 the matrix $\gamma = \gamma^0$ in the chiral representation is hermitian. Otherwise, we proceed as indicated in Remark 1.

The gauge-covariant theory of the free spinor field $\psi \in \Psi$ of the mass m_0, defined on M_n, is based on the gauge invariant action

$$\mathcal{S}[e, \psi] = \int d^n x \sqrt{g} \left\{ \frac{i}{2} [\overline{\psi} \gamma^{\hat\alpha} \nabla_{\hat\alpha} \psi - (\overline{\nabla_{\hat\alpha} \psi}) \gamma^{\hat\alpha} \psi] - m_0 \overline{\psi} \psi \right\} , \tag{16}$$

where $g = |\det(g_{\mu\nu})|$ and $\nabla_\mu = \hat e^{\hat\alpha}_\mu \nabla_{\hat\alpha} = \tilde\nabla_\mu + \Gamma^{spin}_\mu$ are the covariant derivatives formed by the usual ones, $\tilde\nabla_\mu$ (acting in natural indices), and the spin connection

$$\Gamma^{spin}_\mu = \frac{i}{2} e^\beta_{\hat\nu} (\hat e^{\hat\sigma}_\alpha \Gamma^\alpha_{\beta\mu} - \hat e^{\hat\sigma}_{\beta,\mu}) S^{\hat\nu \cdot}_{\cdot \hat\sigma} , \tag{17}$$

giving $\nabla_\mu \psi = (\partial_\mu + \Gamma^{spin}_\mu)\psi$. The action (16) produces the Dirac equation $D\psi = m_0 \psi$ involving the *standard* Dirac operator that can be expressed in terms of point-dependent Dirac matrices as

$$D = i\gamma^\mu \nabla_\mu , \quad \gamma^\mu(x) = e^\mu_{\hat\alpha}(x) \gamma^{\hat\alpha} . \tag{18}$$

Now we can convince ourselves that our definition of the generalized Dirac adjoint is correct since $\overline{\gamma^\mu} = \gamma^\mu$ and $\overline{\Gamma}^{spin}_\mu = -\Gamma^{spin}_\mu$ such that the Dirac operator results to be self-adjoint, $\overline{D} = D$. Moreover, the quantity $\overline{\psi}\psi$ has to be derived as a scalar, i.e. $\nabla_\mu(\overline{\psi}\psi) = \overline{\nabla_\mu \psi}\, \psi + \overline{\psi}\, \nabla_\mu \psi = \partial_\mu(\overline{\psi}\psi)$, while the quantities $\overline{\psi}\gamma^\alpha\gamma^\beta ... \psi$ behave as tensors of different ranks.

Thus we reproduced the main features of the familiar tetrad gauge covariant theories with spin in (1+3)-dimensions from which we can take over all the results arising from similar formulas. In this way we find that the point-dependent matrices $\gamma^\mu(x)$ and $S^{\mu\nu}(x) = e^\mu_{\hat\alpha}(x) e^\nu_{\hat\beta}(x) S^{\hat\alpha\hat\beta}$ have similar properties as (2), (6), (7) and (8), but written in natural indices and with $g(x)$ instead of the flat metric. Using this algebra and the standard notations for the Riemann-Christoffel curvature tensor,

$R_{\alpha\beta\mu\nu}$, Ricci tensor, $R_{\alpha\beta} = R_{\alpha\mu\beta\nu}g^{\mu\nu}$, and scalar curvature, $R = R_{\mu\nu}g^{\mu\nu}$, we recover the useful formulas

$$\nabla_\mu(\gamma^\nu\psi) = \gamma^\nu\nabla_\mu\psi\,, \tag{19}$$

$$[\nabla_\mu,\, \nabla_\nu]\psi = \tfrac{1}{4}R_{\alpha\beta\mu\nu}\gamma^\alpha\gamma^\beta\psi\,, \tag{20}$$

and the identity $R_{\alpha\beta\mu\nu}\gamma^\beta\gamma^\mu\gamma^\nu = -2R_{\alpha\nu}\gamma^\nu$ that allow one to calculate

$$D^2 = -\nabla^2 + \tfrac{1}{4}R\,\mathbf{1}\,, \quad \nabla^2 = g^{\mu\nu}\nabla_\mu\nabla_\nu\,. \tag{21}$$

It remains to complete the operator algebra with new observables from which we have to select complete sets of commuting observables for defining quantum modes.

Another important problem is to find the conserved quantities associated with the symmetries of the theory arising from the action (16). The internal symmetries of the Lagrangian density for $m_0 \neq 0$ reduce to the abelian unitary transformations

$$\psi \to \psi' = U(\xi_{em})\psi = e^{-i\xi_{em}}\psi\,, \quad \xi_{em} \in \mathbb{R}\,, \tag{22}$$

of the group $U(1)_{em}$. Whenever $m_0 = 0$ a supplementary symmetry is given by the transformations of the *chiral* group $U(1)_{ch}$ that read

$$\psi \to \psi' = U(\xi_{ch})\psi = \begin{cases} e^{-i\xi_{ch}\gamma^{ch}}\psi & \text{if} \quad \epsilon_{ch} = -1 \\ e^{\xi_{ch}\gamma^{ch}}\psi & \text{if} \quad \epsilon_{ch} = 1 \end{cases}\,, \tag{23}$$

depending on the real parameters ξ_{ch}. Here it is crucial the operators $U(\xi_{ch})$ be self-adjoint since then $\overline{U}(\xi_{ch})\gamma^\mu U(\xi_{ch}) = \gamma^\mu$ and the Lagrangian density remains invariant.

Theorem 3 (Noether) *The $U(1)_{em}$ internal symmetry produces the vector current*

$$\mathfrak{I}^\mu_{vect} = \overline{\psi}\gamma^\mu\psi\,, \tag{24}$$

that is conserved obeying $\nabla_\mu\mathfrak{I}^\mu_{vect} = 0$. For $m_0 = 0$ the chiral symmetry leads to the conservation of the axial *current,*

$$\mathfrak{I}^\mu_{ax} = \overline{\psi}\gamma^\mu\gamma^{ch}\psi\,, \quad \nabla_\mu\mathfrak{I}^\mu_{ax} = 0\,. \tag{25}$$

Proof: In both cases one starts with infinitesimal transformations and uses the standard method. ■

Thus the notion of conservation of the vector currents gets the same meaning as in special relativity. This give us the possibility to use the Stokes's theorem for defining specific conserved *charges* [21]. Indeed, supposing that $M_n = M_{d+1}$ is a physical spacetime (with $x^0 = t$) and $\sigma(t)$ is a Cauchy surface (i.e. space volumes of dimension d), we can define the *electric* and respectively *chiral* time-independent charges as

$$\mathfrak{Q}_{em} = \int_\sigma d^dx\,\sqrt{g}\,\overline{\psi}\gamma^0\psi\,, \quad \mathfrak{Q}_{ch} = \int_\sigma d^dx\,\sqrt{g}\,\overline{\psi}\gamma^0\gamma^{ch}\psi\,. \tag{26}$$

This result justifies the definition of the time-independent relativistic scalar product of the space Ψ of the spinors defined on M_{d+1} [27, 21].

Definition 5 *In physical spacetimes M_{d+1} the relativistic scalar product $\langle\ ,\ \rangle : \Psi\times\Psi\to\mathbb{C}$ is*

$$\langle\psi,\psi'\rangle=\int_\sigma d^dx\,\sqrt{g}\,\overline{\psi}\gamma^0\psi'\,. \tag{27}$$

According to this definition we can write $\mathfrak{Q}_{em}=\langle\psi,\psi\rangle$ and $\mathfrak{Q}_{ch}=\left\langle\psi,\gamma^{ch}\psi\right\rangle$, opening thus the way to the physical interpretation of the relativistic quantum mechanics that is the starting point to the canonical quantization of the spinor field.

Other conserved current and conserved charges arise from the external symmetries corresponding to the isometries of M_n that will be studied in the next sections.

2.3 The gauge and relativistic covariance

The use of the covariant derivatives assures the covariance of the whole theory under the gauge transformations,

$$\hat{e}^{\hat\alpha}_\mu(x)\ \to\ \hat{e}'^{\hat\alpha}_\mu(x)=\Lambda^{\hat\alpha\cdot}_{\cdot\hat\beta}[A(x)]\,\hat{e}^{\hat\beta}_\mu(x)\,, \tag{28}$$

$$e^\mu_{\hat\alpha}(x)\ \to\ e'^\mu_{\hat\alpha}(x)=\Lambda^{\cdot\hat\beta}_{\hat\alpha\cdot}[A(x)]\,e^\mu_{\hat\beta}(x)\,, \tag{29}$$

$$\psi(x)\ \to\ \psi'(x)=T[A(x)]\,\psi(x)\,, \tag{30}$$

produced by the mappings $A: M_n\to\boldsymbol{G}(\eta)$ the values of which are *local* transformations $A(x)=[\omega(x)]\in\boldsymbol{G}(\eta)$ determined by the set of *real* functions $\omega_{\hat\mu\hat\nu}=-\omega_{\hat\nu\hat\mu}$ defined on M_n. In other words A denotes sections of the spinor fiber bundle that can be organized as a group, $\mathcal{G}(M_n)$, with respect to the multiplication $\times$ defined as $(A'\times A)(x)=A'(x)A(x)$. We use the notations Id for the mapping identity, $Id(x)=1\in\boldsymbol{G}(\eta)$, and A^{-1} for the inverse of A which satisfies $(A^{-1})(x)=[A(x)]^{-1}$.

The general gauge-covariant theory of Dirac spinors outlined here must be also covariant under general coordinate transformation of M_n which, in the *passive* mode, [1] can be seen as changes of the local charts corresponding to the same domain of M_n [23, 24]. If x and x' are the coordinates of a given point in two different charts then there is a mapping ϕ between these charts giving the coordinate transformation $x\to x'=\phi(x)$. These transformations form the group $\mathcal{G}_\phi(M_n)$ with respect to the composition of mappings, $\circ$, defined as usual, i.e. $(\phi'\circ\phi)(x)=\phi'[\phi(x)]$. We denote the identity map of this group by *id* and the inverse mapping of ϕ by ϕ^{-1}.

The coordinate transformations change all the components carrying natural indices including those of the gauge fields [22] changing thus the positions of the local frames with respect to the natural ones. If we assume that the physical experiment makes reference to the axes of the local frame then it could appear situations when several correction of the positions of these frames should be needed before (or after)

[1] We prefer the term of coordinate transformation instead of diffeomorphism since we adopt this viewpoint.

a general coordinate transformation. Obviously, these have to be made with the help of suitable gauge transformations associated to the coordinate ones.

Definition 6 *The* combined *transformation* $(A,\ \phi)$ *is the gauge transformation given by the section* $A \in \mathcal{G}(M_n)$ *followed by the coordinate transformation* $\phi \in \mathcal{G}_\phi(M_n)$.

In this new notation the pure gauge transformations appear as $(A, id) \in \mathcal{G}(M_n)$ while the coordinate transformations will be denoted from now by $(Id, \phi) \in \mathcal{G}_\phi(M_n)$. The effect of a combined transformation (A, ϕ) upon our basic elements is, $x \to x' = \phi(x)$,

$$\psi(x) \to \psi'(x') = T[A(x)]\psi(x)\,, \tag{31}$$

and $e(x) \to \hat{e}'(x')$ where e' are the transformed gauge fields of the components

$$e'^{\mu}_{\hat\alpha}[\phi(x)] = \Lambda^{\cdot\,\hat\beta}_{\hat\alpha\,\cdot}[A(x)]\, e^{\nu}_{\hat\beta}(x)\frac{\partial\phi^\mu(x)}{\partial x^\nu} \tag{32}$$

while the components of $\hat{e}'$ have to be calculated according to Eqs. (12). Thus we have written down the most general transformation laws that leave the action invariant in the sense that $\mathcal{S}[\psi', e'] = \mathcal{S}[\psi, e]$.

The association among the transformations of the gauge group and coordinate transformation leads to a new group with a specific multiplication. In order to find how looks this new operation it is convenient to use the composition among the mappings A and ϕ (taken only in this order) giving the new mappings $A \circ \phi$ defined as $(A \circ \phi)(x) = A[\phi(x)]$. The calculation rules $Id \circ \phi = Id$, $A \circ id = A$ and $(A' \times A) \circ \phi = (A' \circ \phi) \times (A \circ \phi)$ are obvious. In this context one can demonstrate

Theorem 4 *The set of combined transformations of* M_n, $\tilde{\mathcal{G}}(M_n)$, *form a group with respect to the multiplication* $*$ *defined as*

$$(A', \phi') * (A, \phi) = \left((A' \circ \phi) \times A, \phi' \circ \phi\right)\ . \tag{33}$$

Proof: First of all we observe that the operation $*$ is well-defined and represents the composition among the combined transformations since these can be expressed, according to their definition, as $(A, \phi) = (Id, \phi) * (A, id)$. Furthermore, one can verify the result calculating the effect of this product upon the field ψ. ■

Now the identity is (Id, id) while the inverse of a pair (A, ϕ) reads

$$(A, \phi)^{-1} = (A^{-1} \circ \phi^{-1}, \phi^{-1})\,. \tag{34}$$

In addition, one can demonstrate that the group of combined transformations is the semidirect product $\tilde{\mathcal{G}}(M_n) = \mathcal{G}(M_n) \circledS \mathcal{G}_\phi(M_n)$ between the group of sections which is the invariant subgroup and that of coordinate transformations [3]. The same construction starting with the group $\boldsymbol{G}_c(\eta)$ instead of $\boldsymbol{G}(\eta)$ yields the complexified group of combined transformations, $\tilde{\mathcal{G}}_c(M_n)$.

The use of the combined transformations is justified only in theories where there are physical reasons to use some local frames since in natural frames the effect of the combined transformations on the vector and tensor fields reduces to that of their coordinate transformations. However, the physical systems involving spinors can be described exclusively in local frames where our theory is essential. Therein, the vector representation $vect[\tilde{\mathcal{G}}(M_n)]$ is the usual one [22, 21].

Definition 7 *The spinor representation of $\tilde{\mathcal{G}}(M_n)$ has values in the space of the linear operators $U : \Psi \to \Psi$ such that for each (A, ϕ) there exists an operator $U(A, \phi) \in spin[\tilde{\mathcal{G}}(M_n)]$ having the action*

$$U(A, \phi)\psi = [T(A)\psi] \circ \phi^{-1} = [T(A \circ \phi^{-1})](\psi \circ \phi^{-1})\,. \tag{35}$$

This rule gives the transformations (31) in each point of M_n if we put $\psi' = U(A, \phi)\psi$ and then calculate the value of ψ' in the point x'. The Dirac operator derived from $\mathcal{S}$ *covariantly* transforms as

$$(A, \phi) \;\; : \;\; D(x) \to D'(x') = T[A(x)]D(x)\overline{T}[A(x)]\,, \tag{36}$$

where $D' = U(A, \phi)D[U(A, \phi)]^{-1}$. In general, the combined transformations change the form of the Dirac operator which depends on the gauge one uses ($D' \neq D$). We note that for the gauge transformations with $\phi = id$ (when $x' = x$) the action of $U(A, id)$ reduces to the linear transformation given by the matrix $T(A)$.

2.4 Isometries and the external symmetry

In general, the symmetry of the manifold M_n is given by its isometry group, $I(M_n) \subset \mathcal{G}_\phi(M_n)$, whose transformations, $x \to x'(x)$, are coordinate transformation which leave the metric tensor invariant in any chart [22, 23, 24],

$$g_{\alpha\beta}(x')\frac{\partial x'^\alpha}{\partial x^\mu}\frac{\partial x'^\beta}{\partial x^\nu} = g_{\mu\nu}(x)\,. \tag{37}$$

The isometry group is formed by sets of coordinate transformations, $x \to x' = \phi_\xi(x)$, depending on N independent real parameters, ξ^a $(a, b, c... = 1, 2, ..., N)$, such that $\xi = 0$ corresponds to the identity map, $\phi_{\xi=0} = id$. These transformations form a Lie group equipped with the composition rule

$$\phi_{\xi'} \circ \phi_\xi = \phi_{p(\xi',\xi)}\,, \tag{38}$$

where the functions p define the group multiplication. These satisfy $p^a(0, \xi) = p^a(\xi, 0) = \xi^a$ and $p^a(\xi^{-1}, \xi) = p^a(\xi, \xi^{-1}) = 0$ where ξ^{-1} are the parameters of the inverse mapping of ϕ_ξ, $\phi_{\xi^{-1}} = \phi_\xi^{-1}$. Moreover, the structure constants, c_{abc}, of this group can be calculated in the usual way [28],

$$c_{abc} = \left(\frac{\partial p^c(\xi, \xi')}{\partial \xi^a \partial \xi'^b} - \frac{\partial p^c(\xi, \xi')}{\partial \xi^b \partial \xi'^a}\right)\bigg|_{\xi=\xi'=0}\,. \tag{39}$$

These define the commutation relations of the basis generators of the Lie algebra of $I(M_n)$, denoted from now by $i(M_n)$. For small values of the group parameters the infinitesimal transformations, $x^\mu \to x'^\mu = x^\mu + \xi^a k_a^\mu(x) + \cdots$, are given by the Killing vectors k_a whose components,

$$k_a^\mu = \left.\frac{\partial \phi_\xi^\mu}{\partial \xi^a}\right|_{\xi=0}, \tag{40}$$

satisfy the Killing equations $k_{a\,(\mu;\nu)} \equiv k_{a\,\mu;\nu} + k_{a\,\nu;\mu} = 0$ and the identities

$$k_a^\mu k_{b,\mu}^\nu - k_b^\mu k_{a,\mu}^\nu + c_{abc} k_c^\nu = 0\,. \tag{41}$$

The simplest representation of $I(M_n)$ is the *natural* one carried by the space of the *scalar* fields ϑ which transform as $\vartheta \to \vartheta' = \vartheta \circ \phi_\xi^{-1}$. This rule defines the operator-valued representation of the group $I(M_n)$ generated by the operators,

$$L_a = -i k_a^\mu \partial_\mu\,, \quad a = 1, 2, ..., N\,, \tag{42}$$

which are completely determined by the Killing vectors. From Eq. (41) we see that they obey the commutation rules

$$[L_a, L_b] = i c_{abc} L_c\,, \tag{43}$$

given by the structure constants of the Lie algebra $i(M_n)$.

In the theories involving fields with spin, an isometry can change the relative positions of the local frames with respect to the natural ones. This fact may be an impediment when one intends to study the symmetries of these theories in local frames. For this reason it is natural to suppose that the good symmetry transformations we need are isometries preceded by appropriate gauge transformations which should assure that not only the form of the metric tensor would be conserved but the form of the gauge field components too. However, these transformations are nothing other than *particular* combined transformations whose coordinate transformations are isometries.

Definition 8 *The* external symmetry *transformations, (A_ξ, ϕ_ξ), are particular combined transformations involving isometries, $(Id, \phi_\xi) \in I(M_n)$, and corresponding gauge transformations, $(A_\xi, id) \in \mathcal{G}(M_n)$, necessary to* preserve the gauge.

This requirement is accomplished only if we assume that, for given gauge fields e and $\hat{e}$, A_ξ is defined by

$$\Lambda^{\hat\alpha\cdot}_{\cdot\hat\beta}[A_\xi(x)] = \hat{e}^{\hat\alpha}_\mu[\phi_\xi(x)] \frac{\partial \phi_\xi^\mu(x)}{\partial x^\nu}\, e^\nu_{\hat\beta}(x)\,, \tag{44}$$

with the supplementary condition $A_{\xi=0}(x) = 1 \in \boldsymbol{G}(\eta)$. Since ϕ_ξ is an isometry Eq. (37) guarantees that $\Lambda[A_\xi(x)] \in vect[\boldsymbol{G}(\eta)]$ and, implicitly, $A_\xi(x) \in \boldsymbol{G}(\eta)$. Then the transformation laws of our fields are

$$(A_\xi, \phi_\xi): \qquad \begin{array}{rcrcl} x & \to & x' & = & \phi_\xi(x) \\ e(x) & \to & e'(x') & = & e[\phi_\xi(x)] \\ \hat{e}(x) & \to & \hat{e}'(x') & = & \hat{e}[\phi_\xi(x)] \\ \psi(x) & \to & \psi'(x') & = & T[A_\xi(x)]\psi(x)\,. \end{array} \tag{45}$$

The mean virtue of these transformations is that they leave *invariant* the form of the Dirac operator, $D' = D$.

Theorem 5 *The set of the external symmetry transformations (A_ξ, ϕ_ξ) form the Lie group $S(M_n) \subset \tilde{\mathcal{G}}(M_n)$ with respect to the operation $*$. This group, will be called the group of the external symmetry of M_n.*

Proof: Starting with Eq. (44) after a little calculation we find that

$$(A_{\xi'} \circ \phi_\xi) \times A_\xi = A_{p(\xi',\xi)}\,, \tag{46}$$

and, according to Eqs. (33) and (38), we obtain

$$(A_{\xi'}, \phi_{\xi'}) * (A_\xi, \phi_\xi) = (A_{p(\xi',\xi)}, \phi_{p(\xi',\xi)})\,, \tag{47}$$

and $(A_{\xi=0}, \phi_{\xi=0}) = (Id, id)$. ■

From Eq. (47) we understand that $S(M_n)$ is *locally isomorphic* with $I(M_n)$ and, therefore, the Lie algebra $s(M_n)$ of of the group $S(M_n)$ is isomorphic with $i(M_n)$ having the same structure constants. There are arguments that the group $S(M_n)$ must be isomorphic with the universal covering group of $I(M_n)$ since it has anyway the topology induced by $\boldsymbol{G}(\eta)$ which is simply connected. In general, the number of group parameters of $I(M_n)$ or $S(M_n)$ (which is equal to the number of the independent Killing vectors of M_n) can be $0 \le N \le \frac{1}{2}n(n+1)$ [22].

2.5 The spinor representation of $S(M_n)$

The last of Eqs. (45) giving the transformation law of the field ψ defines the operator-valued representation $(A_\xi, \phi_\xi) \to U_\xi$ of the group $S(M_n)$,

$$(U_\xi\psi)[\phi_\xi(x)] = T[A_\xi(x)]\psi(x)\,, \tag{48}$$

which is the spinor representation $spin[S(M_n)] \subset spin[\tilde{\mathcal{G}}(M_n)]$ of the group $S(M_n)$. This representation has unitary transformation matrices in the sense of the Dirac adjoint ($\overline{T} = T^{-1}$) and its transformations leaves the operator D invariant,

$$U_\xi D U_\xi^{-1} = D\,. \tag{49}$$

Since $A_\xi(x) \in \boldsymbol{G}(\eta)$ we say that $spin[S(M_n)]$ is *induced* by the representation $spin[\boldsymbol{G}(\eta)]$ [29, 30].

Theorem 6 *The basis generators of the spinor representation $spin[s(M_n)]$ of the Lie algebra $s(M_n)$ are*

$$X_a = i\frac{\partial U_\xi}{\partial \xi^a}\bigg|_{\xi=0} = L_a + S_a = L_a + \frac{1}{2}\,\Omega_a^{\hat\alpha\hat\beta}S_{\hat\alpha\hat\beta}\,, \tag{50}$$

where

$$\Omega_a^{\hat\alpha\hat\beta} = \left(\hat e^{\hat\alpha}_\mu\, k^\mu_{a,\nu} + \hat e^{\hat\alpha}_{\nu,\mu}k^\mu_a\right) e^\nu_{\hat\lambda}\eta^{\hat\lambda\hat\beta}\,. \tag{51}$$

Proof: For small values of ξ^a, the covariant parameters ω of the element $A_\xi(x) \equiv [\omega_\xi(x)] \in \boldsymbol{G}(\eta)$ can be written as $\omega_\xi^{\hat\alpha\hat\beta}(x) = \xi^a\Omega_a^{\hat\alpha\hat\beta}(x) + \cdots$. Then, from Eq. (44) we can calculate the quantities

$$S_a(x) = i\frac{\partial A_\xi(x)}{\partial \xi^a}\bigg|_{\xi=0} = \frac{1}{2}\,\Omega_a^{\hat\alpha\hat\beta}(x)S_{\hat\alpha\hat\beta}\,, \quad \Omega_a^{\hat\alpha\hat\beta} \equiv \frac{\partial \omega_\xi^{\hat\alpha\hat\beta}}{\partial \xi^a}\Bigg|_{\xi=0}\,, \tag{52}$$

which yields the desired result. ■

We must specify that the functions $\Omega_a^{\hat\alpha\hat\beta}$ are antisymmetric if and only if k_a are Killing vectors. This indicates that the association among isometries and the gauge transformations defined by Eq. (44) is correct.

Remark 2 *The generators (50) can be written in* covariant *form as*

$$X_a = -ik^\mu_a\nabla_\mu + \frac{1}{2}\,k_{a\,\mu;\nu}\,e^\mu_{\hat\alpha}\,e^\nu_{\hat\beta}\,S^{\hat\alpha\hat\beta}\,. \tag{53}$$

In Ref. [3] we have shown that similar formula can be written for any spin, generalizing thus the important result derived in Ref. [1] for the Dirac field in M_4.

Theorem 7 *The operators (50) are self-adjoint with respect to the Dirac adjoint and satisfy the commutation rules*

$$[X_a, X_b] = ic_{abc}X_c\,, \quad [D, X_a] = 0\,, \quad a, b... = 1, 2, ..., N\,, \tag{54}$$

where c_{abc} are the structure constants of the isomorphic Lie algebras $s(M_n) \sim i(M_n)$.

Proof: First we observe that $X_a = \overline{X}_a$ are self-adjoint since all the gamma-matrices have this property. Furthermore, in order to demonstrate Eqs. (54), we derive Eq. (46) with respect to ξ and ξ' and from Eqs. (39) and (51), after a few manipulations, we obtain the identities

$$\eta_{\hat\alpha\hat\beta}\left(\Omega_a^{\hat\alpha\hat\mu}\Omega_b^{\hat\beta\hat\nu} - \Omega_b^{\hat\alpha\hat\mu}\Omega_a^{\hat\beta\hat\nu}\right) + k^\mu_a\Omega^{\hat\mu\hat\nu}_{b,\mu} - k^\mu_b\Omega^{\hat\mu\hat\nu}_{a,\mu} + c_{abc}\Omega_c^{\hat\mu\hat\nu} = 0\,, \tag{55}$$

leading to

$$[S_a, S_b] + [L_a, S_b] - [L_b, S_a] = ic_{abc}S_c\,, \tag{56}$$

and, according to Eq. (43), we find the expected commutation rules. The commutators with the operator D result from Eq. (49). ■
The natural consequence is

Corollary 1 *The operators $U_\xi \in spin[S(M_n)]$ transform the basis generators X_a according to the adjoint representation of $S(M_n)$,*

$$U_\xi X_a U_\xi^{-1} = Adj(\xi)_{ab} X_b \,, \tag{57}$$

defined as

$$Adj(\xi) = e^{i\xi^a adj(X_a)} \,, \quad adj(X_a)_{bc} = -ic_{abc} \,, \tag{58}$$

where $adj(X_a)$ are the basis generators of the adjoint representation of $s(M_n)$.

Proof: This is a general result of the group representation theory [29]. We note that here the phase factors are chosen such that the commutators

$$[adj(X_a),\, adj(X_b)] = ic_{abc}\, adj(X_c) \tag{59}$$

keep the form (54). ■
Whenever the field ψ obeys convenient conditions at the boundary of σ then these operators are Hermitian with respect to the relativistic scalar product (27) and the representation $spin[S(M_n)]$ is *unitary* (with $X_a^\dagger = X_a$ and $U^\dagger = U^{-1}$). In this case one can define quantum modes correctly, using the set of commuting operators formed by the Casimir operators of $spin[s(M_n)]$, the generators of its Cartan subalgebra and the Dirac operator, D.

The non-covariant form (50) of the generators X_a helps us to understand the meaning of the notion of *manifest* covariance in curved manifolds where the representations of $S(M_n)$ are induced by those of $\boldsymbol{G}(\eta)$, depending thus on the gauge fixing. For this reason, in general, their generators have point-dependent spin terms that do not commute with the orbital parts. However, one may find several special gauge fixings where some spin terms become point-independent.

Definition 9 *When the generators $S_a(x)$, $a = 1, 2, ..., N'$ $(N' \le N)$, of a subgroup $G_1 \subset S(M_n)$ are independent on x obeying $[S_a,\, L_b] = 0$, for all $a = 1, 2, ..., N'$ and $b = 1, 2, ..., N$, we say that ψ behaves manifestly covariant with respect to this subgroup.*

The point-independent operators S_a, $a = 1, 2, ..., N'$, are then just the generators of an usual linear representation of G_1. One knows many examples of curved spacetimes for which one can choose suitable local frames where the spinor fields transform manifestly covariant with respect to different subgroups of $S(M_n)$ or even to this whole group. Particularly, the frames of the flat spacetimes where the fields with spin transform manifestly covariant under the transformations of the $S(M_n)$ group are nothing other than the usual *inertial* frames of the special relativity.

From the physical point of view, our approach is useful since this allows one to derive the conserved quantities predicted by the Noether theorem.

Theorem 8 (Noether) *The basis generators $X_a \in spin[s(M_n)]$ give rise to conserved currents, $\mathfrak{J}^\mu[X_a]$, which satisfy*

$$\mathfrak{J}^\mu[X_a]_{;\mu} = 0\,. \tag{60}$$

Proof: The conserved currents have to be calculated from the action (16) in the usual way, starting with the infinitesimal transformations generated by X_a. One finds

$$\mathfrak{J}^\mu[X_a] = -\frac{i}{2}\left[\overline{\psi}\gamma^{\hat\alpha}e^\mu_{\hat\alpha}\partial_\nu\psi - (\overline{\partial_\nu\psi})\gamma^{\hat\alpha}e^\mu_{\hat\alpha}\psi\right]k^\nu_a + \frac{1}{4}\,\overline{\psi}\{\gamma^{\hat\alpha}, S^{\hat\beta\hat\gamma}\}\psi\, e^\mu_{\hat\alpha}\,\Omega_{a\,\hat\beta\hat\gamma}\,, \tag{61}$$

where ψ satisfies the Dirac equation. Notice that the first term here involves a part of the stress-energy tensor of the Dirac field [22, 27]. ■

Corollary 2 *If $M_n = M_{d+1}$ is a physical spacetime then every basis generator X_a defines its specific time-independent quantity*

$$\mathfrak{Q}_a = \int_\sigma d^dx\,\sqrt{g}\,\mathfrak{J}^0[X_a] = \frac{1}{2}\left(\langle\psi, X_a\psi\rangle + \langle X_a\psi, \psi\rangle\right)\,. \tag{62}$$

Proof: Bearing in mind that the time coordinate of Definitions 3 and 5 was $t = x^0$ and using Eqs. (61) and (50), one can arrange the terms in order to obtain this formula. ■

Whenever the operators X_a are Hermitian with respect to the relativistic scalar product (27) one can write $\mathfrak{Q}_a = \langle\psi, X_a\psi\rangle$. In the relativistic quantum mechanics this quantity has to be interpreted as the expectation value of the observable X_a in the state described by the spinor ψ. Of course, at the level of the quantum field theory $\mathfrak{Q}_a$ becomes the one-particle operator which takes over the role of the generator X_a [4].

Hence we have built a complete theory of the external symmetries related to the genuine isometries defined as coordinate transformations of M_n which preserve the metric tensor. This is very close to the theory of the Poincaré group of the Minkowski spacetime, producing conserved quantities through the Noether theorem in a similar manner as in special relativity.

3 Dirac-type operators related to K-Y tensors

Our theory of the external symmetry is not suitable for the study of other types of symmetries having more subtle geometrical origins as the so called hidden symmetries encapsulated in the existence of the S-K and K-Y tensors. In the classical theory, the hidden symmetries are arising from more general isometries defined in the whole phase space which cannot be reduced to pure coordinate transformations. For this reason the previous group theoretical methods seem to be not appropriate for obtaining new conserved quantities or operators commuting with D, produced by the S-K or K-Y tensors fields at the quantum level. Here new specific mechanisms have to be exploited for analysing the hidden symmetries or several new types of supersymmetries.

3.1 Operators produced by S-K and K-Y tensors

It is obvious that in the classical theory only the S-K tensors, $\tilde{k}^{(r)}$, can give rise directly to new conserved quantities since these are completely symmetric tensors of a given rank, $r \geq 2$, whose components, $\tilde{k}^{(r)}_{\mu_1\mu_2...\mu_r}$, satisfy the Killing equation, $\tilde{k}^{(r)}_{(\mu_1\mu_2...\mu_r;\mu)=0}$. They allow one to construct the quantities $\tilde{k}^{(r)}_{\mu_1\mu_2...\mu_r}\dot{x}^{\mu_1}\dot{x}^{\mu_2}...\dot{x}^{\mu_r}$ that are conserved along the geodesics. Unfortunately, this property does not hold in the quantum theory because of the *quantum anomaly* grace of which, in manifolds with non-vanishing Ricci tensor, the operators $K^{(r)} = \tilde{k}^{(r)\,\mu_1\mu_2...\mu_r}\nabla_{\mu_1}\nabla_{\mu_2}...\nabla_{\mu_r}$ do not commute with the usual Laplace operator of M_n, ∇^2, as it might be expected. Particularly, in the case of the second order operators one finds [31]

$$\left[K^{(2)}, \nabla^2\right] = \frac{4}{3}\left(\tilde{k}^{(2)\,\mu\,[\nu}R^{\sigma]\,\cdot}_{\cdot\;\nu}\right)_{;\sigma}\nabla_\mu\,. \tag{63}$$

However, this inconvenience is given up at least in the case of the second rank S-K tensors that can be expressed as symmetrized contractions of K-Y tensors. On the other hand, the K-Y tensors, that were first introduced from purely mathematical reasons [5], are profoundly connected to the supersymmetric classical and quantum mechanics on curved manifolds where such tensors do exist [32]. Thus it seems that, at least from the point of view of the quantum theory of (super)symmetry, the natural generalization of the Killing vectors are the K-Y tensors rather than the S-K ones.

The K-Y tensors, $\tilde{f}^{(r)}$, are completely skew-symmetric tensors of rank r for which the Killing equation reads

$$\tilde{f}^{(r)}_{\mu_1\mu_2...(\mu_r;\mu)} \equiv \tilde{f}^{(r)}_{\mu_1\mu_2...\mu_r;\mu} + \tilde{f}^{(r)}_{\mu_1\mu_2...\mu;\mu_r} = 0\,. \tag{64}$$

It was surprising to see that the K-Y tensors are naturally related to the Dirac theory in curved manifolds since all of them are able to produce first-order differential operators which commutes or anticommutes with D. The Killing vectors considered K-Y tensors of rank $r = 1$ give rise to the operators X_a defined by Eq. (50). This result was reported in [1] simultaneously with the operators built using second rank K-Y tensors [33, 34]. A recent generalization [6] yields

Theorem 9 *Given a K-Y tensor $\tilde{f}^{(r)}$ of an arbitrary rank $r = 1, 2, ...$, the operator*

$$\begin{aligned} Y[\tilde{f}^{(r)}] \;\; = \;\; & (-1)^r i\gamma^{\mu_1}\gamma^{\mu_2}\cdots\gamma^{\mu_{r-1}}\left(\tilde{f}^{(r)}_{\mu_1\mu_2...\mu_{r-1}\cdot}{}^{\mu_r}\nabla_{\mu_r}\right.\\ & \left. -\frac{1}{2(r+1)}\tilde{f}^{(r)}_{\mu_1\mu_2...\mu_r;\mu}\gamma^{\mu_r}\gamma^{\mu}\right) \end{aligned} \tag{65}$$

commute with D if r is odd and anticommute with D if r is even.

Proof: We delegate the proof to the Ref. [6]. ■
In general, one can construct new operators commuting with D using the operators (65) built with the help of arbitrary K-Y tensors. Indeed, given two K-Y tensors of any rank, $\tilde{f}^{(r_1)}$ and $\tilde{f}^{(r_2)}$, the new second order operator $K^{(2)} = \{Y[\tilde{f}^{(r_1)}], Y[\tilde{f}^{(r_2)}]\}$ commutes with D whenever $r_1 + r_2$ is an even number. Moreover, in this way we obtain the corresponding factorized S-K tensor of the second rank that gives rise to the operator $K^{(2)}$ freely of quantum anomaly. In this manner one can generate new types of operators that help one to investigate the hidden symmetries and to obtain large sets of conserved operators that may constitute new (super)algebras. In other respects, the implication of the K-Y tensors in the quantum theory suggests us that such tensors with complex-valued components would be also useful even if from the classical viewpoint these are pointless.

Of a particular interest are the operators built with the help of the second rank K-Y tensors, $\tilde{f}$, with real or complex-valued components $\tilde{f}_{\mu\nu} = -\tilde{f}_{\nu\mu}$ which satisfies the equation (64) for $r = 2$.

Definition 10 *The operators*

$$D_{\tilde{f}} = i\gamma^{\mu}\left(\tilde{f}_{\mu\cdot}^{\cdot\,\nu}\nabla_{\nu} - \tfrac{1}{6}\tilde{f}_{\mu\nu;\rho}\gamma^{\nu}\gamma^{\rho}\right), \tag{66}$$

given by the second rank K-Y tensors, $\tilde{f}$, are called Dirac-type operators.

These are non-standard Dirac operators which obey $\{D_{\tilde{f}}, D\} = 0$ and can be involved in new types of genuine or hidden (super)symmetries. Remarkable superalgebras of Dirac-type operators can be produced by special second-order K-Y tensors that represent square roots of the metric tensor.

3.2 Roots and their Dirac-type operators

Let us start with some technical details and the basic definitions. Given ρ an arbitrary tensor field of rank 2 defined on a domain of M_n, we denote with the same symbol $\langle\rho\rangle$ the equivalent matrices with the elements $\rho^{\mu\,\cdot}_{\cdot\,\nu}$ in natural frames and $\rho^{\hat{\alpha}\,\cdot}_{\cdot\,\hat{\beta}} = \hat{e}^{\hat{\alpha}}_{\mu}\rho^{\mu\,\cdot}_{\cdot\,\nu}e^{\nu}_{\hat{\beta}}$ in local frames. We say that ρ is non-singular on M_n if $\det\langle\rho\rangle \neq 0$ on a domain of M_n where the metric is non-singular. This tensor is said irreducible on M_n if its matrix is irreducible.

Definition 11 *The non-singular real or complex-valued K-Y tensor f of rank 2 defined on M_n which satisfies*

$$f^{\mu\,\cdot}_{\cdot\,\alpha}f_{\mu\beta} = g_{\alpha\beta}\,, \tag{67}$$

is called an unit root of the metric tensor of M_n, or simply an unit root of M_n.

It was shown that any K-Y tensor that satisfy Eq. (67) is covariantly constant [9],

$$f_{\mu\nu;\sigma}=0\,. \tag{68}$$

Since Eq. (67) can be written as $f^{\mu\cdot}_{\cdot\alpha}f^{\alpha\cdot}_{\cdot\nu}=-\delta^{\mu}_{\nu}$ this takes the matrix form

$$\langle f\rangle^2=-1_n\,, \tag{69}$$

where the notation 1_n stands for the $n\times n$ identity matrix. Hereby we see that the unit roots are matrix representations of several *complex units* (similar to $i\in\mathbb{C}$) with usual properties as, for example, $\langle f\rangle^{-1}=-\langle f\rangle$. The unit roots having only *real*-valued components are called *complex structures* and represent automorphisms of the tangent fiber bundle $\mathcal{T}(M_n)$ of M_n. In local frames these appear as particular point-dependent transformations of the gauge group $G(\eta)=vect[\boldsymbol{G}(\eta)]$. The manifold possessing such structures are said to have a Kählerian geometry (see the Appendix A). However, the unit roots considered here are beyond this case since these are defined as automorphisms of the complexified fiber bundle $\mathcal{T}(M_n)\otimes\mathbb{C}$, being thus transformations of the complexified group $G_c(\eta)=vect[\boldsymbol{G}_c(\eta)]$.

As in the case of the complex structures of the Kählerian geometries, the matrices of the unit roots have specific algebraic properties resulted from Eq. (69). These can be pointed out in local frames (where the matrix elements are $f^{\hat\alpha\cdot}_{\cdot\hat\beta}=\hat e^{\hat\alpha}_{\mu}f^{\mu\cdot}_{\cdot\nu}e^{\nu}_{\hat\beta}$) using gauge transformations of $G(\eta)$.

Lemma 1 *The matrix of any root of M_n is equivalent with a matrix completely reducible in 2×2 diagonal blocks.*

Proof: The matrix $\langle f\rangle$ which satisfies Eq. (69) has only two-dimensional invariant subspaces spanned by pairs of vectors z and $\langle f\rangle z$. On these subspaces, Eq. (69) is solved in local frames by two types of 2×2 unimodular blocks without diagonal elements: either skew-symmetric blocks with factors ± 1, when the involved dimensions are of the same signature, or symmetric ones with pure imaginary phase factors, $\pm i$, if the signatures are opposite. However, the diagonalization procedure cannot be continued using transformations of $G(\eta)$ since these preserve the form of the 2×2 blocks which are proportional with the generators of the subgroups $SO(2)$ or $SO(1,1)$ acting on the corresponding invariant subspaces. Notice that other transformations of $G_c(\eta)$ are not useful since these do not leave the metric invariant. ∎

This selects the geometries allowing unit roots.

Corollary 3 *The unit roots are allowed only by manifolds M_n with an even number of dimensions, $n=2k$, $k\le l$.*

Proof: If n is odd then the 2×2 blocks do not cover all dimensions, so that $\det\langle\hat f\rangle=0$ and f is no more an unit root of M_n. ∎

Corollary 4 *The unit roots of M_n have real matrices only when the metric η has a signature with even n_+ and n_-. Otherwise the unit roots have only complex-valued matrices. In both cases the matrices of the unit roots are unimodular, i.e.* $\det\langle f\rangle = 1$.

It is clear that for the real-valued unit roots (i.e., complex structures) one can construct the *symplectic* 2-forms $\tilde{\omega} = \frac{1}{2} f_{\mu\nu} dx^\mu \wedge dx^\nu$ which are closed and non-degenerate.

The above properties indicate that the unit roots are defined up to sign. Therefore, if two unit roots f_1 and f_2 do not obey the condition $f_1 = \pm f_2$ then these will be considered *different* between themselves. We denote by $\boldsymbol{R}_1(M_n)$ the set of all different unit roots of the manifold M_n. On the other hand, when an unit root f is multiplied by an arbitrary *real* number $\alpha \neq 0$, we say that $\rho(x) = \alpha f(x)$ is a *root* of norm $\|\rho\| = |\alpha|$. Thus we can associate to any unit root f the one-dimensional linear real space $L_f = \{\rho \,|\, \rho = \alpha f,\, \alpha \in \mathbb{R}\}$ in which each non vanishing element is a root. According to Corollary 4, when the metric η is pseudo-Euclidean, the unit roots can have complex matrix elements and in that case the unit root f and its *adjoint*, f^*, are different. This last one generates its own linear real space L_{f^*} of adjoint roots which satisfy $[\,\langle\rho\rangle^*\,,\,\langle\rho'\rangle\,] = 0$, $\forall\rho,\, \rho' \in L_f$ since the matrices of f and f^* commutes with each other, having same diagonal blocks up to signs.

The whole set of roots of M_n defined as

$$\boldsymbol{R}(M_n) = \bigcup_{f\in \boldsymbol{R}_1(M_n)} (L_f - \{0\}) \tag{70}$$

seems to have special algebraic structure since it does not have the element zero and, in general, it is not certain that a linear combination of roots is a root too. To convince this, it is enough to observe that the sum of the roots ρ and ρ^* is no more a root since $\det(\langle\rho\rangle + \langle\rho\rangle^*) = 0$ when $\rho^* \neq \rho$ because of the reduction of the pure imaginary diagonal 2×2 blocks. Moreover, the product of the matrices of two different roots gives a nonsingular matrix but that may be not of a root. Thus we understand that $\boldsymbol{R}(M_n) + \{0\}$ cannot be organized as a global linear space or algebra even though, according to the definition (70), it naturally includes linear parts as L_f or L_{f^*}. In other respects, we know examples indicating that $\boldsymbol{R}(M_n)$ may contain subsets which are parts of some linear spaces with one or three dimensions, isomorphic with Lie algebras [12, 10]. In any event, the algebraic properties and the topology of $\boldsymbol{R}(M_n)$ seem to be complicated depending on the topological structure of the set of unit roots $\boldsymbol{R}_1(M_n) \subset \boldsymbol{R}(M_n)$.

The K-Y tensor gives rise to Dirac-type operators of the form (66) which have an important property formulated in [9].

Theorem 10 *The Dirac-type operator D_f produced by the K-Y tensor f satisfies the condition*

$$(D_f)^2 = D^2 \tag{71}$$

if and only if f is an unit root.

Proof: The arguments of Ref. [9] show that the condition Eq. (71) is equivalent with Eqs. (67) and (68). Moreover, we note that for $f \in \boldsymbol{R}_1(M_n)$ the square of the Dirac-type operator

$$D_f = i f^{\cdot\,\nu}_{\mu\cdot} \gamma^\mu \nabla_\nu \,, \tag{72}$$

has to be calculated exploiting the identity $0 = f_{\mu\nu;\alpha;\beta} - f_{\mu\nu;\beta;\alpha} = f_{\mu\sigma} R^{\sigma}_{\cdot\,\nu\alpha\beta} + f_{\sigma\nu} R^{\sigma}_{\cdot\,\mu\alpha\beta}$, which gives

$$R_{\mu\nu\alpha\beta} f^{\mu\cdot}_{\cdot\,\sigma} f^{\nu\cdot}_{\cdot\,\tau} = R_{\sigma\tau\alpha\beta} \tag{73}$$

and leads to Eq. (71). ■

Thus we conclude that the equivalence of the condition (71) with Eqs. (67) and (68) holds in any geometry of dimension $n = 2k$ allowing roots. When $f^* \neq f$ then D_{f^*} is different from D_f even if $(D_f)^2 = (D_{f^*})^2 = D^2$. These operators are no longer self-adjoint, obeying $\overline{D_f} = D_{f^*}$ and

$$\{D_f,\, D\} = 0\,, \quad \{D_{f^*},\, D\} = 0\,. \tag{74}$$

3.3 Continuous symmetries generated by unit roots

Now we shall reach an interesting points of our study, showing that there are continuous transformations able to relate the operators D_f and D to each other. We know that in many particular cases [12, 10, 11] this is possible and now we intend to point out that this is a general property of theories involving roots. To this end we introduce a new useful point-dependent matrix.

Definition 12 *Given the unit root f, the matrix*

$$\Sigma_f = \frac{1}{2} f_{\mu\nu} S^{\mu\nu} \tag{75}$$

is the spin-like operator associated to f.

This is a matrix that acts on the space of spinors Ψ and, therefore, can be interpreted as a generator of the spinor representation $spin[\boldsymbol{G}_c(\eta)]$ since the components of f are, in general, complex-valued functions. It has the obvious property $\overline{\Sigma_f} = \Sigma_{f^*}$ while from (19) and (67) one obtains that it is covariantly constant in the sense that $\nabla_\nu(\Sigma_f \psi) = \Sigma_f \nabla_\nu \psi$. Hereby we find that the Dirac-type operator (72) can be written as

$$D_f = i\,[D,\, \Sigma_f]\;, \tag{76}$$

where D is the standard Dirac operator defined by Eq. (18). Moreover, from Eqs. (74) we deduce $[\Sigma_f,\, D^2] = [\Sigma_f,\, (D_f)^2] = 0$ and similarly for Σ_{f^*}.

Definition 13 *We say that $G_f = \{(A_\rho, id)\,|\,\rho = \alpha f,\, \alpha \in \mathbb{R}\} \subset \tilde{\mathcal{G}}_c(M_n)$ is the one-parameter Lie group associated to the unit root $f \in \boldsymbol{R}_1(M_n)$.*

The spinor representation of this group, $spin(G_f)$, is the restriction to G_f of the representation $spin[\tilde{\mathcal{G}}_c(M_n)]$. Therefore, the operators $U_\rho \in spin(G_f)$, have the action

$$(U_\rho\psi)(x) = [T(\rho)\psi](x) = T[\alpha f(x)]\,\psi(x)\,, \tag{77}$$

where the transformation matrices

$$T(\alpha f) = e^{-i\alpha\Sigma_f} \in spin(\boldsymbol{G}_c) \tag{78}$$

depend on the group parameter $\alpha \in \mathbb{R}$. Hence we defined the new mappings $A_\rho : M_n \to \boldsymbol{G}_c(\eta)$ representing sections of the complexified spinor fiber bundle such that $A_\rho(x) = [\rho(x)] \in \boldsymbol{G}_c(\eta)$. Since the matrices (78) are just those defined by Eq. (9) where we replace ω by the roots $\rho = \alpha f \in L_f$, their action on the point-dependent Dirac matrices results from Eq. (10) to be,

$$[T(\alpha f)]^{-1}\gamma^\mu T(\alpha f) = \Lambda^{\mu\cdot}_{\cdot\nu}(\alpha f)\gamma^\nu\,, \tag{79}$$

where $\Lambda^{\mu\cdot}_{\cdot\nu} = e^\mu_{\hat\alpha}\Lambda^{\hat\alpha\cdot}_{\cdot\hat\beta}\hat e^{\hat\beta}_\nu$ are matrix elements with natural indices of the matrix

$$\Lambda(\alpha f) = e^{\alpha\langle f\rangle} = 1_n\cos\alpha + \langle f\rangle\sin\alpha\,, \tag{80}$$

calculated according to Eqs. (11) and (69). We note that this is a matrix representation of the usual *Euler formula* of the complex numbers. Now it is obvious that in local frames $\langle f\rangle = \Lambda(\frac{\pi}{2}f) \in vect(\boldsymbol{G}_c)$, as mentioned above.

Theorem 11 *The operators $U_\rho \in spin(G_f)$, with $\rho = \alpha f$, have the following action in the linear space spanned by the operators D and D_f:*

$$U_\rho D(U_\rho)^{-1} = T(\alpha f)D[T(\alpha f)]^{-1} = D\cos\alpha + D_f\sin\alpha\,, \tag{81}$$
$$U_\rho D_f(U_\rho)^{-1} = T(\alpha f)D_f[T(\alpha f)]^{-1} = -D\sin\alpha + D_f\cos\alpha\,. \tag{82}$$

Proof: From Eq. (80) we obtain the matrix elements $\Lambda^{\mu\cdot}_{\cdot\nu}(\alpha f) = \cos\alpha\,\delta^\mu_\nu + \sin\alpha\, f^{\mu\cdot}_{\cdot\nu}$ which lead to the above result since Σ_f as well as $T(\alpha f)$ are covariantly constant. ∎
From this theorem it results that $\alpha \in [0, 2\pi]$ and, consequently, the group $G_f \sim U(1)$ is *compact*. Therefore, it must be a subgroup of the maximal compact subgroup of $\boldsymbol{G}_c$. In addition, from Eq. (80) we see that $L_f \sim so(2)$ is the Lie algebra of the vector representation of G_f that is the compact group $vect(G_f) = \{\Lambda(\alpha f)|\alpha \in [0, 2\pi]\} \sim U(1)$. Note that the transformations (81) and (82) leave invariant the operator $D^2 = (D_f)^2$ because this commutes with the spin-like operator Σ_f which generates these transformations.

Particularly, if M_n allows real-valued unit roots (i.e. complex structures) this is an usual Kähler manifold. In general, when f has complex components (and $f^* \neq f$) then $L_{f^*} \sim so(2)$ is a different linear space representing the Lie algebra of $vect(G_{f^*})$. These two Lie algebras are complex conjugated to each other but remain isomorphic since they are real algebras. The relation among the transformation matrices of $spin(G_f)$ and $spin(G_{f^*})$ is $\overline{T}(\alpha f) = T(-\alpha f^*) = [T(\alpha f^*)]^{-1}$ which means that when $f^* \neq f$ the representation $spin(G_f)$ is no more unitary in the sense of the generalized Dirac adjoint.

The conclusion is that an unit root gives rise simultaneously to a Dirac-type operator D_f which satisfies Eq. (71) and the one-parameter Lie group G_f one needs to relate D and D_f to each other.

3.4 Symmetries due to families of unit roots

The next step is to investigate if there could appear higher symmetries given by non-abelian Lie groups with many parameters, embedding different abelian groups G_f produced by some sets of unit roots which have to form bases of linear spaces isomorphic with the Lie algebras of these non-abelian groups. Such Lie algebras must include many one-dimensional Lie algebras L_f being thus subsets of $\boldsymbol{R}(M_n) + \{0\}$ where we know that the linear properties are rather exceptions. Therefore, we must look for special *families* of unit roots, $\boldsymbol{f} = \{f^i \,|\, i = 1, 2, ..., N_f\} \subset \boldsymbol{R}_1(M_n)$, having supplementary properties which should guarantee simultaneously that: (I) the linear space $L_{\boldsymbol{f}} = \{\rho \,|\, \rho = \rho_i f^i,\, \rho_i \in \mathbb{R}\}$ is isomorphic with a real Lie algebra, and (II) each element $\rho \in L_{\boldsymbol{f}} - \{0\}$ is a root (of an arbitrary norm).

The first condition is accomplished only if the set $\{T(\rho) \,|\, \rho \in L_{\boldsymbol{f}}\}$ includes a Lie group with N_f parameters. This means that the operators $\Sigma^i = \Sigma_{f^i}$, $i = 1, 2, ...N_f$ must be (up to constant factors) the basis-generators of a Lie algebra with some real structure constants c_{ijk}. Then according to Eqs. (8) and (75), we can write

$$\left[\Sigma^i, \Sigma^j\right] = \tfrac{i}{2}\left[\,\langle f^i\rangle\,, \langle f^j\rangle\,\right]_{\mu\nu} S^{\mu\nu} = i c_{ijk}\Sigma^k\,, \tag{83}$$

obtaining a necessary condition for $\boldsymbol{f}$ be a family of unit roots,

$$[\,\langle f^i\rangle\,, \langle f^j\rangle\,] = c_{ijk}\langle\, f^k\,\rangle\,. \tag{84}$$

The condition (II) is accomplished only when $\langle\rho\rangle^2$ is equal up to a positive factor (i.e. the squared norm) with -1_n. This requires to have

$$\{\,\langle f^i\rangle\,, \langle f^j\rangle\,\} = -2\kappa_{ij} 1_n\,. \tag{85}$$

where κ is a *positive definite* metric that can be brought in canonical form $\kappa_{ij} = \delta_{ij}$ through a suitable choice of the unit roots. If $\boldsymbol{f}$ satisfy simultaneously Eqs. (84) and (85) then $L_{\boldsymbol{f}}$ is just the Lie algebra of the group $\{\Lambda(\rho) \,|\, \rho \in L_{\boldsymbol{f}}\}$ the matrices of which read

$$\Lambda(\rho) = e^{\rho_i\langle f^i\rangle} = 1_n \cos\|\rho\| + \nu_i\,\langle f^i\rangle \sin\|\rho\|\,, \tag{86}$$

where $\|\rho\| = \sqrt{\rho_i\rho_i}$ (when we take $\kappa_{ij} = \delta_{ij}$) and $\nu_i = \rho_i/\|\rho\|$. All these results lead to the following

Theorem 12 *If the set $\boldsymbol{f} = \{f^i \,|\, i = 1, 2, ..., N_f\} \in \boldsymbol{R}_1(M_n)$ is a family of unit roots then the matrices 1_n and $\langle f^i \rangle$, $i = 1, 2, ..., N_f$, form the basis of a matrix representation of a finite-dimensional associative algebra over $\mathbb{R}$.*

Proof: Since $\boldsymbol{f}$ is a family of unit roots in the sense of above definition, f^i must satisfy Eqs. (84) and (85) with the canonical metric. Hereby it results that the set of the real linear combinations $\rho_0 1_n + \rho_i \langle f^i \rangle$ forms an associative algebra with respect to the matrix multiplication that can be calculated by adding the commutator and anticommutator. Moreover, this algebra is a division one since there exists the zero element (with $\rho_0 = 0$, $\rho_i = 0$), the unit element is 1_n and each element different from zero has the inverse $(\rho_0 1_n + \rho_i \langle f^i \rangle)^{-1} = (\rho_0 1_n - \rho_i \langle f^i \rangle)/({\rho_0}^2 + \rho_i \rho_i)$. Obviously, this real algebra is finite possessing a basis of dimension $N_f + 1$ where $\langle f^i \rangle$ play the role of complex units. Eq. (86) can be interpreted as a matrix representation of the Euler formula. ■

This theorem severely restricts the existence of the families of unit roots. Indeed, according to the Frobenius theorem there are only two finite real algebras able to give suitable representations in spaces of roots, namely the algebra $\mathbb{C}$ of complex numbers and the *quaternion* algebra, $\mathbb{H}$. In the first case we have *isolated* unit roots f and representations of the $\mathbb{C}$ algebra generated by the matrices 1_n and $\langle f \rangle$ (which play the role of $i \in \mathbb{C}$) related to the continuous symmetry group $G_f \sim U(1)$ we studied in the previous section.

Here we focus on the second possibility leading to families of unit roots with $N_f = 3$ that constitute matrix representations of the quaternion units.

Theorem 13 *The unique type of family of unit roots with $N_f > 1$ having the properties (I) and (II) are the triplets $\boldsymbol{f} = \{f^1, f^2, f^3\} \subset \boldsymbol{R}_1(M_n)$ which satisfy*

$$\langle f^i \rangle \, \langle f^j \rangle = -\delta_{ij} 1_n + \varepsilon_{ijk} \langle f^k \rangle \,, \quad i, j, k... = 1, 2, 3 \,. \tag{87}$$

Proof: Taking into account that ε_{ijk} is the anti-symmetric tensor with $\varepsilon_{123} = 1$ we recognize that Eqs. (87) are the well-known multiplication rules of the quaternion units or similar algebraic structures (e.g. the Pauli matrices). Consequently, the matrices $\langle f^i \rangle$ and 1_n generate a matrix representation of $\mathbb{H}$. Other choices are forbidden by the Frobenius theorem. ■

If the unit roots f^i have only real-valued components we recover the *hypercomplex structures* defining hyper-Kähler geometries (presented in the Appendix A).

Eqs. (87) combined with the previous results (83)-(86) provide all the features of the specific continuous symmetry associated to $\boldsymbol{f}$.

Definition 14 *We say that $G_{\boldsymbol{f}} = \{(A_\rho, id) \,|\, \rho \in L_{\boldsymbol{f}}\} \sim SU(2) \subset \boldsymbol{G}_c(\eta)$ is the Lie group associated to the triplet $\boldsymbol{f} \subset \boldsymbol{R}_1(M_n)$.*

The spinor and the vector representations of this group are determined by the representations of its Lie algebra, $g_{\boldsymbol{f}}$, resulted from Theorem 13.

Corollary 5 *The basis-generators of* $vect(g_{\boldsymbol{f}}) = L_{\boldsymbol{f}}$ *are* $\frac{i}{2} f^i$ *while the basis-generators of the algebra* $spin(g_{\boldsymbol{f}}) \sim su(2)$ *read* $\hat{s}_i = \frac{1}{2}\Sigma^i$ $(i = 1, 2, 3)$.

Proof: From Eqs. (87) and (84) we deduce that $c_{ijk} = 2\varepsilon_{ijk}$. Furthermore, from Eqs. (83) and (84) we obtain the standard commutation rules of $SU(2)$ generators,

$$[\hat{s}_i,\, \hat{s}_j] = i\varepsilon_{ijk}\hat{s}_k\,, \tag{88}$$

and similarly for $\frac{i}{2}\langle f^i\rangle$. ∎
Now $vect(G_{\boldsymbol{f}}) = \{\Lambda(A_\rho)\,|\,\rho \in L_{\boldsymbol{f}},\, \|\rho\| \le 2\pi\}$ is the compact group formed by the matrices $\Lambda(A_\rho) = \Lambda(\rho)$ of the form (86) constructed using the elements of the Lie algebra $L_{\boldsymbol{f}} \sim su(2) \sim so(3)$. The transformation matrices giving the action of the operators $U_\rho \in spin(G_{\boldsymbol{f}})$,

$$T(\rho) = e^{-i\rho_i\Sigma^i} = e^{-2i\rho_i\hat{s}_i}\,, \quad \rho = \rho_i f^i \in L_{\boldsymbol{f}}, \tag{89}$$

have to be calculated directly from Eq. (78) replacing $\alpha = \pm\|\rho\| = \pm\sqrt{\rho_i\rho_i}$ and $f = \pm\rho/\|\rho\|$. Then the transformations (79) can be expressed in terms of the parameters ρ_i using the matrices $\Lambda(\rho)$. Hereby we observe that ρ_i are nothing other than the analogous of the well-known Cayley-Klein parameters but ranging in a larger spherical domain (where $\|\rho\| \le 2\pi$) such that they cover two times the group $G_{\boldsymbol{f}} \sim SU(2)$, as we can convince ourselves calculating

$$\Lambda(\rho)\,\langle f^i\rangle\,\Lambda^T(\rho) = \mathfrak{R}_{ij}(2\vec{\rho})\,\langle f^j\rangle\,, \quad \forall\,\Lambda_\rho \in vect(G_{\boldsymbol{f}})\,, \tag{90}$$

where $\mathfrak{R}(2\vec{\rho}) \in O(3)$ is the rotation of the Cayley-Klein parameters $2\rho_i$. These arguments lead to the conclusion that $vect(G_{\boldsymbol{f}}) \sim SU(2)$ [12]. On the other hand, since the rotations (90) change the basis of $L_{\boldsymbol{f}}$ leaving Eqs. (87) invariant, we understand that these form the group $Aut(L_{\boldsymbol{f}})$, of the automorphisms of the Lie algebra $L_{\boldsymbol{f}}$ considered as a real algebra.

In the case of triplets involving only real-valued unit roots when the geometry is hyper-Kähler, every family of real unit roots (i.e., a hypercomplex structure) $\boldsymbol{f}$ has its own Lie algebra $L_{\boldsymbol{f}} \sim su(2)$. These algebras cannot be embedded in a larger one because of the restrictions imposed by the Frobenius theorem. An example of hyper-Kähler manifold is the Euclidean Taub-NUT space which is equipped with only one family of real unit roots [12, 11]. The manifolds with pseudo-Euclidean metric with odd n_+ and n_- have only pairs of *adjoint* triplets, $\boldsymbol{f}$ and $\boldsymbol{f}^*$, the last one being formed by the adjoints of the unit roots of $\boldsymbol{f}$. The spaces $L_{\boldsymbol{f}}$ and $L_{\boldsymbol{f}^*}$ are isomorphic between themselves (as real vector spaces) and all the results concerning the symmetries generated by $\boldsymbol{f}^*$ can be taken over from those of $\boldsymbol{f}$ using complex conjugation. Moreover, we must specify that the set $L_{\boldsymbol{f}} \bigcup L_{\boldsymbol{f}^*}$ is no more a linear space since the linear operations among the elements of $L_{\boldsymbol{f}}$ and $L_{\boldsymbol{f}^*}$ are not allowed.

An example is the Minkowski spacetime which has a pair of conjugated triplets of complex-valued unit roots [10]. Both these examples of manifolds possessing triplets with the properties (87) are of dimension four. The results we know indicate that similar properties may have other manifolds of dimension $n = 4k$, $k = 1, 2, 3, ...$ where we expect to find many such triplets [35]. The main geometric feature of all these manifolds is given by

Theorem 14 *If a manifold M_n allows a triplet of unit roots then this must be Ricci flat (having $R_{\mu\nu} = 0$).*

Proof: As in the case of any hyper-Kähler manifold, using Eqs. (73) and (87) we calculate the expression $R_{\mu\nu\alpha\beta} f^{1\,\alpha\beta} = R_{\mu\nu\sigma\beta} f^{3\,\sigma\cdot}_{\ \ \cdot\alpha} (\langle f^3 \rangle \langle f^1 \rangle)^{\alpha\beta} = R_{\mu\nu\sigma\beta} f^{3\,\sigma\cdot}_{\ \ \cdot\alpha} f^{2\,\alpha\beta} = -R_{\mu\nu\alpha\beta} f^{1\,\alpha\beta}$ which vanishes. Furthermore, permutating the first three indices of R we find the identity

$$2R_{\mu\alpha\nu\beta} f^{1\,\alpha\beta} = R_{\mu\nu\alpha\beta} f^{1\,\alpha\beta} = 0\,. \tag{91}$$

Finally, writing $R_{\mu\nu} = R_{\mu\alpha\nu\beta} f^{1\,\alpha\cdot}_{\ \ \cdot\tau} f^{1\,\beta\tau} = -R_{\mu\alpha\sigma\beta} f^{1\,\sigma\cdot}_{\ \ \cdot\nu} f^{1\,\alpha\beta} = 0$, we draw the conclusion that the manifold is Ricci flat. The same procedure holds for f^2 or f^3 leading to identities similar to (91). Note that the manifolds possessing only single unit roots (as the Kähler ones) are not forced to be Ricci flat. ∎

Starting with a triplet $\boldsymbol{f} = \{f^1, f^2, f^3\} \subset \boldsymbol{R}_1(M_n)$ satisfying (87) one can construct a rich set of Dirac-type operators of the form $D(\vec{\nu}) = \nu_i D^i$ where $\vec{\nu}$ is an unit vector (with $\vec{\nu}^2 = 1$) and $D^i = D_{f^i} = i[D, \Sigma^i]$, $i = 1, 2, 3$, play the role of a *basis*. This set is *compact* and isomorphic with the sphere of unit roots $S^2_{\boldsymbol{f}} = \{f_{\vec{\nu}} \,|\, f_{\vec{\nu}} = \nu_i f^i,\ \vec{\nu}^2 = 1\} \subset L_{\boldsymbol{f}}$, since $D(\vec{\nu}) = D_{f_{\vec{\nu}}}$ for any $f_{\vec{\nu}} \in S^2_{\boldsymbol{f}}$. Moreover, each operator $D(\vec{\nu})$ can be related to D through the transformations (81) and (82) of the one-parameter subgroup $G_{f_{\vec{\nu}}} \subset G_{\boldsymbol{f}} \sim SU(2)$ defined by $f_{\vec{\nu}}$.

Theorem 15 *The operators $U_\rho \in spin(G_{\boldsymbol{f}})$ transform the Dirac operators D, D^i ($i = 1, 2, 3$) as*

$$U_\rho D [U_\rho]^{-1} = T(\rho) D [T(\rho)]^{-1} = D \cos\|\rho\| + \nu_i D^i \sin\|\rho\|\,, \tag{92}$$

$$\begin{aligned} U_\rho D^i [U_\rho]^{-1} &= T(\rho) D^i [T(\rho)]^{-1} \\ &= D^i \cos\|\rho\| - (\nu_i D + \varepsilon_{ijk} \nu_j D^k) \sin\|\rho\|\,, \end{aligned} \tag{93}$$

where $\rho = \rho_i f^i = \|\rho\| \nu_i f^i$.

Proof: We consider the result of Theorem 11 for each one-parameter subgroup of $G_{\boldsymbol{f}}$ generated by the unit roots $f_{\vec{\nu}} = \nu_i f^i$. The straightforward calculation starting with Eq. (89) is also efficient. ∎

The previous results indicate that the set $\boldsymbol{R}_1(M_n)$, of unit roots producing Dirac-type operators, has an interesting topological structure involving either single f producing isolated Dirac-type operators or unit spheres $S^2_{\boldsymbol{f}}$ leading to compact sets of Dirac-type operators. In order to show off this structure one needs to exploit the mechanisms of our theory based on the fact that the linear spaces L_f or $L_{\boldsymbol{f}}$ are isomorphic with the Lie algebras of the symmetry groups of the Dirac-type operators generated by spin-like operators.

In the non-Kählerian manifolds equipped with pairs of adjoint triplets $\boldsymbol{f}$ and $\boldsymbol{f}^*$, the Dirac-type operators D^i and $D_{(f^i)^*} = \overline{D}^i$ are related to each other through the Dirac adjoint. However, when an extended symmetry dealing with the physical needs would be necessary, we may consider the complexified group $(G_{\boldsymbol{f}})_c \sim SL(2,\mathbb{C})$ of $G_{\boldsymbol{f}}$. Then we have to work with more complicated groups and Lie algebras since the complexification doubles the number of generators of the spinor or vector representations. For example, the generators of the complexified spinor representation, $spin(g_{\boldsymbol{f}})_c$, are $\hat{s}_i$ and $(\pm)i\hat{s}_i$ and similarly for $vect(g_{\boldsymbol{f}})_c$.

3.5 Supersymmetry and isometries in Kählerian manifolds

Beside the types of continuous symmetries we have studied, the presence of the unit roots gives rise to supersymmetries related to the external symmetries in an interesting manner. In order to avoid the complications due to the presence of the pair of adjoint triplets we restrict ourselves to discuss in this section only Kählerian manifolds.

In a Kähler manifold, a complex structure $f = f^*$ generates its own $\mathcal{N} = 2$ *real* superalgebra, $\boldsymbol{d}_f = \{D(\lambda)\,|\,D(\lambda) = \lambda_0 D + \lambda_1 D_f\}$ where D and D_f (obeying $\{D, D_f\} = 0$, $(D_f)^2 = D^2$) form a basis. This basis can be changed through the transformations (81) and (82) that preserve the anticommutation relations of the basis operators, forming the group of automorphisms of $\boldsymbol{d}_f$, $Aut(\boldsymbol{d}_f) \sim SO(2)$. If the manifold has a non-trivial isometry group $I(M_n)$ then an arbitrary isometry $x \to x' = \phi_\xi(x)$ transforms f as a second rank tensor,

$$f_{\mu\nu}(x) \to f'_{\mu\nu}(x')\frac{\partial x'^{\mu}}{\partial x_\alpha}\frac{\partial x'^{\nu}}{\partial x_\beta} = f_{\alpha\beta}(x)\,. \tag{94}$$

When there is only one f we are forced to put $f' = f$ which means this remains *invariant* under isometries.

Theorem 16 *If a Kähler manifold M_n with the external symmetry group $S(M_n)$ has a single complex structure, f, then every generator $X \in spin[s(M_n)]$ commutes with D_f.*

Proof: We calculate first the derivatives with respect to ξ^a of Eq. (94) for $f' = f$ and $\xi = 0$. Then, taking into account that f is covariantly constant we can write $f_{\alpha\lambda}k^{\lambda}{}_{;\beta} = f_{\beta\lambda}k^{\lambda}{}_{;\alpha}$ for each Killing vector field k defined by Eq. (40). This identity yields

$$[X,\, \Sigma_f] = 0\,, \quad [X,\, D_f] = 0\,, \quad \forall\, X \in spin[s(M_n)]\,, \tag{95}$$

which means that the operators Σ_f and D_f are invariant under isometries. ∎

The case of the hyper-Kähler manifolds is more complicated since a triplet $\boldsymbol{f}$ gives rise to self-adjoint Dirac-type operators $D^i = \overline{D}^i$ which anticommute with D and present the continuous symmetry discussed in the previous section. In these conditions a new algebraic structure is provided by

Theorem 17 *If a triplet $\boldsymbol{f} \subset \boldsymbol{R}_1(M_n)$ accomplishes Eqs. (87) then the corresponding Dirac-type operators satisfy*

$$\left\{D^i,\, D^j\right\} = 2\delta_{ij}D^2\,, \quad \left\{D^i,\, D\right\} = 0\,. \tag{96}$$

Proof: If $i = j$ we take over the result of Theorem 10. For $i \neq j$ we take into account that M_n is Ricci flat finding that D^i and D^j anticommute. The second relation was demonstrated earlier for any unit root. ∎

Thus it is clear that the operators D and D^i $(i = 1, 2, 3)$ form a basis of a four-dimensional real superalgebra of Dirac operators.

Definition 15 *The set $\boldsymbol{d}_{\boldsymbol{f}} = \{D(\lambda)|D(\lambda) = \lambda_0 D + \lambda_i D^i; \lambda_0, \lambda_i \in \mathbb{R}\}$ is the $\mathcal{N} = 4$ superalgebra generated by the triplet $\boldsymbol{f}$.*

This superalgebra contains the subset $\boldsymbol{d}^1_{\boldsymbol{f}} = \{D(\nu)\,|\,{\nu_0}^2 + \vec{\nu}^2 = 1\}$ of the Dirac operators which have the property $D(\nu)^2 = D^2$. The set $\boldsymbol{d}^1_{\boldsymbol{f}}$ has the topology of the sphere S^3 including the sphere S^2 of the Dirac-type operators $D(\vec{\nu}) = D(0, \vec{\nu})$.

Furthermore, it is natural to study the group of automorphisms of this superalgebra, $Aut(\boldsymbol{d}_{\boldsymbol{f}})$, and its Lie algebra, $aut(\boldsymbol{d}_{\boldsymbol{f}})$. Obviously, these automorphisms have to be linear transformations among D and D^i preserving their anticommutation rules. The transformation matrices $T(\rho)$ commute with D^2, leaving Eqs. (96) invariant under transformations (92) and (93) which appear thus as automorphisms of $\boldsymbol{d}_{\boldsymbol{f}}$. Consequently, the group of these transformations, $vect(G_{\boldsymbol{f}}) \sim SU(2)$, is a subgroup of $Aut(\boldsymbol{d}_{\boldsymbol{f}})$. However, we need more automorphisms in order to complete the group $Aut(\boldsymbol{d}_{\boldsymbol{f}})$ with more ordinary or invariant subgroups, isomorphic with $SU(2)$ or $O(3)$. These supplemental automorphisms must transform the operators D^i among themselves preserving their anticommutators as well as the form of D. Therefore, these may be produced by the transformations of $S(M_n)$ since these leave the operator D invariant.

In what concerns the transformation of the triplets $\boldsymbol{f}$ under isometries we have two possibilities, either to consider that all the complex structures $f^i \in \boldsymbol{f}$ are invariant under isometries or to assume that the isometries transform the components of the triplet among themselves, ${f'}^i = \hat{\mathfrak{R}}_{ij} f^j$, such that Eqs. (87) remain invariant. The first hypothesis is not suitable since we need more transformations in order to fill in the group $Aut(\boldsymbol{d}_{\boldsymbol{f}})$ when we do not have other sources of symmetry. Therefore, we must adopt the second viewpoint assuming that the components of $\boldsymbol{f}$ are transformed as

$$f^j_{\mu\nu}(x')\frac{\partial x'^{\,\mu}}{\partial x_\alpha}\frac{\partial x'^{\,\nu}}{\partial x_\beta}=\hat{\mathfrak{R}}_{kj}(\xi,x)f^k_{\alpha\beta}(x)\,, \tag{97}$$

by 3×3 real *orthogonal* matrices $\hat{\mathfrak{R}}\in O(3)$ that leave Eqs. (87) invariant. Their matrix elements can be put in the equivalent forms

$$\begin{aligned}\hat{\mathfrak{R}}_{ij}(\xi,x) &= \frac{1}{n}f^{i\,\alpha\beta}(x)\frac{\partial\phi^\mu_\xi(x)}{\partial x_\alpha}\frac{\partial\phi^\nu_\xi(x)}{\partial x_\beta}f^j_{\mu\nu}[\phi_\xi(x)]\\ &= \frac{1}{n}f^{i\,\hat\alpha\hat\beta}(x)\Lambda^{\hat\mu\,\cdot}_{\cdot\,\hat\alpha}[A_\xi(x)]\Lambda^{\hat\nu\,\cdot}_{\cdot\,\hat\beta}[A_\xi(x)]f^j_{\hat\mu\hat\nu}[\phi_\xi(x)]\,.\end{aligned} \tag{98}$$

The last formula is suitable for calculations in local frames where we must consider the external symmetry using the gauge transformations $\Lambda[A_\xi(x)]\in vect[\boldsymbol{G}(\eta)]$ defined by Eq. (44) and associated to isometries for preserving the gauge.

The matrices $\hat{\mathfrak{R}}$ might be point-dependent and depend on the parameters ξ^a of $I(M_n)$. This means that the canonical covariant parameters $\hat\omega_{ij}=-\hat\omega_{ji}$ giving the expansion $\hat{\mathfrak{R}}_{ij}(\hat\omega)=\delta_{ij}+\hat\omega_{ij}+\cdots$ are also depending on these variables. Then, for small values of the parameters ξ^a the covariant parameters can be developed as $\hat\omega_{ij}=\xi^a\hat c_{aij}+...$ emphasizing thus the quantities $\hat c_{aij}$ we shall see that do not depend on coordinates.

Theorem 18 *Let M_n be a hyper -Kähler manifold having the hypercomplex structure $\boldsymbol{f}=\{f^1,f^2,f^3\}$ and a non-trivial isometry group $I(M_n)$ with parameters ξ^a and the corresponding Killing vectors k_a as defined by Eq. (40). Then the basis-generators $X_a\in spin[s(M_n)]$ and $\hat s_i=\frac{1}{2}\Sigma^i\in spin(g_{\boldsymbol{f}})\sim su(2)$ satisfy*

$$[X_a,\,\hat s_i]=i\hat c_{aij}\hat s_j\,,\quad a=1,2,...,N\,, \tag{99}$$

where $\hat c_{aij}$ are point-independent *structure constants.*

Proof: Deriving Eq. (97) with respect to ξ^a in $\xi=0$ we deduce

$$f^i_{\mu\lambda}k^\lambda_{a\,;\nu}-f^i_{\nu\lambda}k^\lambda_{a\,;\mu}=\hat c_{aij}f^j_{\mu\nu}\,, \tag{100}$$

which leads to the explicit form

$$\hat c_{aij}=-\frac{2}{n}\,\varepsilon_{ijl}\,\hat k^l_a\,,\quad \hat k^l_a=f^{l\,\mu\nu}k_{a\,\mu;\nu}\,. \tag{101}$$

Bearing in mind that $f^i_{\mu\nu;\sigma}=0$ and using the identity $f^{i\,\mu\nu}k_{a\,\mu;\nu;\sigma}=R_{\mu\sigma\nu\,\cdot}^{\cdot\,\cdot\,\cdot\,\lambda}k_{a\,\lambda}f^{i\,\mu\nu}$ [21] and Eq. (91), we find that $\nabla_\sigma\hat k^i_a=f^{i\,\mu\nu}k_{a\,\mu;\nu;\sigma}=0$ which means that $\partial_\sigma\hat c_{aij}=0$. Finally, from Eq. (100) we derive the commutation rules (99). ■

Now we can point out how act the isometries $x\to x'=\phi_\xi(x)$ on the operators D^i.

Corollary 6 *The Dirac-type operators D^i produced by any triplet $\boldsymbol{f}$ transform under isometries according to the representation $O_{\boldsymbol{f}} = \{\hat{\mathfrak{R}}(\xi) \,|\, (Id, \phi_\xi) \in I(M_n)\}$ of the group $I(M_n)$* induced *by the group $O(3)$. The matrices $\hat{\mathfrak{R}}(\xi)$ of this representation have the form (98) and give the transformation rule*

$$(U_\xi D^i U_\xi^{-1})[\phi_\xi(x)] = [T(\xi) D^i \overline{T}(\xi)](x) = \hat{\mathfrak{R}}_{ij}(\xi) D^j(x)\,, \tag{102}$$

where the action of $U_\xi \in spin[S(M_n)]$ is defined by Eq. (48).

Proof: The matrices (98) generated as any adjoint representation,

$$\hat{\mathfrak{R}}(\xi) = e^{i\xi^a \mathfrak{X}_a}\,, \quad (\mathfrak{X}_a)_{ij} = -i\hat{c}_{aij}\,, \tag{103}$$

are point-independent since $\hat{c}_{aij}$ are structure constants. Moreover, if we commute Eq. (99) with X_b using Eqs. (54) and (88) we obtain $[\mathfrak{X}_a, \mathfrak{X}_b] = ic_{abc}\mathfrak{X}_c$ concluding that $O_{\boldsymbol{f}}$ is a well-defined induced representation of $I(M_n)$. From Eqs. (75) and (97) we derive

$$(U_\xi \Sigma^i U_\xi^{-1})(x') = [T(\xi)\Sigma^i \overline{T}(\xi)](x) = \hat{\mathfrak{R}}_{ij}(\xi)\Sigma^j(x)\,, \tag{104}$$

which leads to Eq. (102) after a commutation with D that is invariant under U_ξ. ■
Hence we have the complete image of the symmetries that preserve the anticommutation rules of the real superalgebra $\boldsymbol{d_f}$ in a hyper-Kähler manifold. These are encapsulated in the group $Aut(\boldsymbol{d_f})$ whose transformations are defined by Eqs. (92), (93) and (102).

Corollary 7 *The group $Aut(\boldsymbol{d_f}) = vect(G_{\boldsymbol{f}}) ⓢ O_{\boldsymbol{f}}$ is a representation of the semidirect product $G_{\boldsymbol{f}}$ ⓢ $S(M_n)$ where $G_{\boldsymbol{f}}$ is the invariant subgroup.*

Proof: The basis generators of the Lie algebra $aut(\boldsymbol{d_f})$ of the group $Aut(\boldsymbol{d_f})$ are the operators $\hat{s}_i$ and X_a that obey the commutation relations (54), (88) and (99). These operators form a Lie algebra since $\hat{c}_{aij}$ are point-independent. In this algebra $g_{\boldsymbol{f}} \sim su(2)$ is an ideal and, therefore, the corresponding $SU(2)$ subgroup is invariant. However, this result can be obtained directly taking $(A_\xi,\, \phi_\xi) \in S(M_n)$ and $(A_\rho, id) \in G_{\boldsymbol{f}}$ and evaluating $(A_\xi,\, \phi_\xi) * (A_\rho,\, id) * (A_\xi,\, \phi_\xi)^{-1} = ([A_\xi \times (A_\rho \times A_\xi^{-1})] \circ \phi_\xi^{-1},\, id) = (A_{\rho'},\, id)$ where, according to (44), (86) and (97), we have $\rho' = \rho_i \hat{\mathfrak{R}}_{ij}(\xi) f^j$. Consequently, $(A_{\rho'},\, id) \in G_{\boldsymbol{f}}$ which means that $G_{\boldsymbol{f}} \sim SU(2)$ is an invariant subgroup. ■
Another important consequence of the previous theorem is

Corollary 8 *The basis generators of $spin[S(M_n)]$ and the Dirac-type operators of the $\mathcal{N} = 4$ superalgebra $\boldsymbol{d_f}$ obey*

$$[X_a, D^i] = i\hat{c}_{aij} D^j\,. \tag{105}$$

Proof: This formula results commuting Eq. (99) with D. ■
Finally, we find an interesting restriction that can be formulated as

Corollary 9 *The minimal condition that M_n allows a hypercomplex structure is to have an isometry group that includes at least one $O(3)$ subgroup.*

Proof: The subgroup $O_{\boldsymbol{f}} \sim O(3)$ of $Aut(\boldsymbol{d}_{\boldsymbol{f}})$ needs at least three generators X_a satisfying the $su(2)$ algebra. Thus we conclude that $S(M_n)$ must include one $SU(2)$ group for each different hypercomplex structure of M_n. ■
This restriction is known in four dimensions where there exists only three hyper-Kähler manifolds with only one hypercomplex structure and one subgroup $O(3) \subset I(M_4)$ [36]. These are given by the Atiyah-Hitchin [37], Taub-NUT and Eguchi-Hanson [38] metrics, the first one being only that does not admit more $U(1)$ isometries [36, 39]. In addition, we have the example of the four-dimensional Euclidean flat space that has two different triplets and the isometry group $E(4)$ including the group $O(4) \sim O(3) \times O(3)$.

The theory above can be easily extended to non-Kählerian manifolds having pairs of adjoint triplets of unit roots. In this case Eq. (98) gives complex-valued orthogonal matrices which oblige us to start with the complex superalgebra $(\boldsymbol{d}_{\boldsymbol{f}})_c$ (defined over $\mathbb{C}$ instead of $\mathbb{R}$) and with the complexified groups and Lie algebras [40]. Then the group of automorphisms of $(\boldsymbol{d}_{\boldsymbol{f}})_c$, will be the group $Aut(\boldsymbol{d}_{\boldsymbol{f}})_c = vect(G_{\boldsymbol{f}})_c \circledS O_{\boldsymbol{f}}$, that is a representation of th group $(G_{\boldsymbol{f}})_c \circledS S(M_n)$ where $O_{\boldsymbol{f}}$ is the representation of the group $S(M_n)$ induced by the group of the complex-valued orthogonal matrices $O(3)_c$. Obviously, the invariant subgroup here is $(G_{\boldsymbol{f}})_c \sim SL(2,\mathbb{C})$. The minimal condition that M_n allows a pair of adjoint triplets is the group $S(M_n)$ to include at least one $SL(2,\mathbb{C})$ subgroup since we need six generators for building the representation $O_{\boldsymbol{f}} \sim O(3)_c$. The Minkowski spacetime which has a pair of adjoint triplets and $O(3,1)$ isometries is a typical example (see the Appendix B).

3.6 Discrete symmetries

In many physical problems the study of the discrete symmetries could be also productive. Of course, the results concerning the continuous symmetries obtained above will be crucial for understanding the structure of the discrete transformations which relate among themselves the standard Dirac operator and the Dirac-type ones [10, 12, 11].

Let us start with the simplest case of an isolated unit root f.

Theorem 19 *For any unit root f there exists the discrete group $\mathbb{Z}_4(f) \subset G_f$ the orbit of which is $\{D, -D, D_f, -D_f\}$.*

Proof: Using Eqs. (81) and (82) one observes that the transformations $\mathbf{1}$, $U_f = T(\frac{\pi}{2}f)$, $\digamma = (U_f)^2 = T(\pi f)$, and $(U_f)^3 = T(-\frac{\pi}{2}f) = \digamma U_f = U_f \digamma$ form the spinor representation of the cyclic group $\mathbb{Z}_4(f)$. Since $\mathcal{F}^2 = \mathbf{1}$, the pair $(\mathbf{1}, \mathcal{F})$ represents the subgroup $\mathbb{Z}_2 \subset \mathbb{Z}_4(f)$. According to Eq. (81) we find that

$$D_f = U_f D (U_f)^{-1} \,, \tag{106}$$

while the action of the matrix $\mathcal{F}$,

$$\mathcal{F}\gamma^\mu \mathcal{F} = -\gamma^\mu \,, \tag{107}$$

is independent on the form of f so that this changes the sign of all the Dirac or Dirac-type operators. ■

For a given manifold, M_n, the matrix $\mathcal{F}$ is uniquely defined up to a factor ± 1. Thus $\mathcal{F}$ is in some sense independent on the discrete symmetry group where is involved, playing the role of a chiral matrix. For this reason it is convenient to identify $\gamma^{ch} = \mathcal{F}$ bearing in mind that then we must have $\overline{\mathcal{F}} = \epsilon_{ch}\mathcal{F} = \pm\mathcal{F}$. When the metric is pseudo-Euclidean then the operators of the spinor representation of the discrete group $\mathbb{Z}_4(f^*)$ produced by $f^* \neq f$ have to be written directly using the Dirac adjoint. Indeed, from Eq. (106) we obtain

$$U_{f^*} = (\overline{U_f})^{-1} = \overline{U_f}\; \overline{\mathcal{F}} \,, \tag{108}$$

and similarly for the other operators. Starting with these elements the remaining operators of the cyclic group will be obtained using multiplication [29].

A most interesting case is that of the discrete symmetries of the Dirac-type operators D^i $(i = 1, 2, 3)$ given by the triplet $\boldsymbol{f}$ which satisfy Eqs. (87) [11].

Theorem 20 *The Dirac operators $\pm D$ and the Dirac-type ones $\pm D^1$, $\pm D^2$, and $\pm D^3$ are related among themselves through the transformations of the spinor representation of the quaternion group, $\mathbb{Q}(\boldsymbol{f}) \subset G_{\boldsymbol{f}}$.*

Proof: Let us denote by $U_i = T(\frac{\pi}{2} f^i)$ the operators that, according to Theorem 19, have the properties

$$(U_1)^2 = (U_2)^2 = (U_3)^2 = \mathcal{F} \,, \quad \mathcal{F}^2 = \mathbf{1} \,, \tag{109}$$

and $\mathcal{F} U_i = U_i \mathcal{F}$. Furthermore, from Eqs. (79), (80) and (87) we deduce

$$U_1 U_2 = U_3 \,, \quad U_2 U_3 = U_1 \,, \qquad U_3 U_1 = U_2 \,, \tag{110}$$

$$U_2 U_1 = U_3 \mathcal{F} \,, \quad U_3 U_2 = U_1 \mathcal{F} \,, \qquad U_1 U_3 = U_2 \mathcal{F} \,, \tag{111}$$

and, after a few manipulation, we see that $\mathbf{1}$, $\mathcal{F}$, U_i and $\mathcal{F} U_i$ $(i = 1, 2, 3)$ form a representation of the dicyclic group $\langle 2, 2, 2 \rangle$ [29] which is isomorphic with the quaternion subgroup of $G_{\boldsymbol{f}}$ we denote by $\mathbb{Q}(\boldsymbol{f})$. Using Eqs. (106) and (107) we find that its orbit in the space of the Dirac operators is the desired one. ■

As expected, the cyclic groups $\mathbb{Z}_4(f^i)$ are subgroups of $\mathbb{Q}(\boldsymbol{f})$. For this reason the spinor representation of the group $\mathbb{Q}(\boldsymbol{f}^*)$ has to be derived from that of $\mathbb{Q}(\boldsymbol{f})$ using the same method as in the case of cyclic groups.

Hence we conclude that for each isolated unit root f one can define a finite group $\mathbb{Z}_4(f)$ which is a subgroup of G_f while the triplets $\boldsymbol{f}$ produce more complicated finite discrete groups, $\mathbb{Q}(\boldsymbol{f}) \subset G_{\boldsymbol{f}}$. Since the groups G_f and $G_{\boldsymbol{f}}$ cannot be embedded in a larger group, the product of two operators of the spinor representation of two different discrete groups is, in general, an arbitrary operator which do not correspond to a transformation of another discrete group $\mathbb{Z}_4$ or $\mathbb{Q}$. In addition, this new operator could transform the standard Dirac operator in a new operator having different properties to those of the Dirac-type ones. Therefore when we restrict ourselves to orbits containing only the Dirac and Dirac-type operators we have to consider only the discrete groups discussed above [11].

Finally we note that in any Dirac theory the discrete symmetries due to the existence of the unit roots appear in association with the transformations of parity (P) and charge conjugation (C). The form of these transformations depends on the physical meaning of the theory as well as on the metric signature. In physical spacetimes M_{d+1}, described by Definition 3, these transformations can be introduced in a similar way as in QED [41]. Thus the parity changes $\boldsymbol{x} \to -\boldsymbol{x}$ and $\psi(\boldsymbol{x}, t) \to \gamma^0\psi(-\boldsymbol{x}, t)$ leaving the Dirac equation invariant. In addition, if there are involved only Dirac free fields on M_{d+1}, we can adopt a type of charge conjugation close to that of QED. The gamma-matrices of M_{d+1} satisfy Eqs. (2) and, consequently, they must be either symmetric or skew-symmetric. Assuming that there are $s+1$ symmetric gamma-matrices, namely γ^0 and s matrices with space-like indices, $\gamma^{\hat\alpha_1}, \gamma^{\hat\alpha_2}, ..., \gamma^{\hat\alpha_s}$, we observe that the matrix $C = (-1)^{\frac{s}{2}}\gamma^0\gamma^{\hat\alpha_1}\gamma^{\hat\alpha_2}, ..., \gamma^{\hat\alpha_s}$ has the convenient properties $C^{-1} = C^T = (-1)^s\overline{C}$ and $C\gamma^{\hat\mu}C^{-1} = (-1)^s(\gamma^{\hat\mu})^T$. With its help one can define the charge conjugated spinor $\psi^c = C\overline{\psi}^T$ of the spinor ψ and verify that the equation of the free Dirac field remains invariant under the charge conjugation $\psi \to \psi^c$. This transformation is point-independent which means that the vacuum state could be *stable* (or invariant [27]) in quantum field theories based on free field equations invariant under this type of charge conjugation. However, the above definition of the charge conjugation does not hold in Kaluza-Klein theories where several space dimensions are used for introducing the interaction.

4 The Euclidean Taub-NUT space

Involved in many modern studies in physics [42], the metric of the Euclidean Taub-NUT space is a self-dual instanton solution with self-dual Riemann tensor [43, 37] of the Euclidean Einstein equations without cosmological constant. The Taub-NUT space is of interest since beside isometries there are hidden symmetries giving rise to conserved quantities associated to S-K tensors [44]. There is a conserved vector, analogous to the Runge-Lenz vector of the Kepler type problem, whose existence is rather surprising in view of the complexity of the equations of motion [45, 46, 47, 48]. These hidden symmetries are related with the existence of four K-Y tensors generating the S-K ones [46, 36, 49, 13, 14]. Three such tensors form in fact a hypercamplex structure giving the hyper-Kählerian character of this geometry.

The quantum theory in the Euclidean Taub-NUT background has also interesting specific features in the case of the scalar fields [48, 19] as well as for fields with spin half where our results [8, 15, 16] complete the previous ones [50]. In both cases there exit large algebras of conserved observables [18] including the components of the angular momentum and three components of the Runge-Lenz operator that lead to six-dimensional dynamical algebras [44, 16, 18]. Remarkably, the orbital angular momentum has a special unusual form that generates new harmonics, called $SO(3) \otimes U(1)$-harmonics [19], and corresponding new spherical spinors [8]. These will enter in the structure of the particular solutions of the Klein-Gordon or Dirac equations giving the discrete quantum modes of the scalar or spin half particles. Moreover, in the Dirac theory in this geometry, beside the new Dirac-type operators there are similar operators as those of the scalar theory but completed with spin terms that help us to understand the spin effects in Kählerian manifolds.

4.1 $SO(3) \otimes U(1)$ isometry transformations

The Euclidean Taub-NUT manifold denoted from now by M is a 4-dimensional Kaluza-Klein space which has static charts with Cartesian coordinates x^μ $(\mu, \nu, ... = 1,2,3,4)$ where x^i $(i,j,... = 1,2,3)$ are the *physical* Cartesian space coordinates while x^4 is the Cartesian extra-coordinate. Taking $\eta = 1_4$ we have to use the three-dimensional vector notations, $\vec{x} = (x^1, x^2, x^3)$, $r = |\vec{x}|$ and $dl^2 = d\vec{x} \cdot d\vec{x}$, for writing the line element

$$ds^2 = \frac{1}{V(r)} dl^2 + V(r)[dx^4 + A_i^{em}(\vec{x})dx^i]^2 \,, \tag{112}$$

defined by the specific functions

$$\frac{1}{V} = 1 + \frac{\mu}{r} \,, \quad A_1^{em} = -\frac{\mu}{r}\frac{x^2}{r+x^3} \,, \quad A_2^{em} = \frac{\mu}{r}\frac{x^1}{r+x^3} \,, \quad A_3^{em} = 0 \,. \tag{113}$$

The real number μ is the main parameter of the theory. If one interprets $\vec{A}^{em}$ as the vector potential (or gauge field) it results the magnetic field with central symmetry

$$\vec{B}^{em} = \mu \frac{\vec{x}}{r^3} \,. \tag{114}$$

Other important charts are those with spherical coordinates $(r, \theta, \varphi, \chi)$ where r, θ, φ, are commonly related to the physical Cartesian ones, x^i, while the fourth coordinate χ is defined by

$$x^4 = -\mu(\chi + \varphi) \,. \tag{115}$$

In this chart the radial coordinate belongs the radial domain D_r where $r > 0$ if $\mu > 0$ or $r > |\mu|$ if $\mu < 0$, while the angular coordinates θ, φ cover the sphere S^2 and $\chi \in D_\chi = [0, 4\pi)$. The line element in spherical coordinates is

$$ds^2 = \frac{1}{V}(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\varphi^2) + \mu^2 V (d\chi + \cos\theta\, d\varphi)^2\,, \tag{116}$$

since

$$A_r^{em} = A_\theta^{em} = 0\,, \quad A_\varphi^{em} = \mu(1 - \cos\theta)\,. \tag{117}$$

The Euclidean Taub-NUT space possesses a special type of isometries which combines the space transformations with the gauge transformations of the gauge field $\vec{A}^{em}(\vec{x})$. There are $U(1)_4$ translations $x^4 \to x'^4 = x^4 + a^4$ which leave the metric invariant if a^4 is a point-independent real constant. Moreover, if one takes $a^4 = a^4(\vec{x})$ an arbitrary function of $\vec{x}$ then these become gauge transformations preserving the form of the line element only if one requires $\vec{A}^{em}$ to transform as

$$A_i^{em}(\vec{x}) \to A_i'^{\,em}(\vec{x}) = A_i^{em}(\vec{x}) - \partial_i a^4(\vec{x})\,. \tag{118}$$

Thus it is obvious that $U(1)_4$ is an isometry group playing, in addition, the role of the gauge group associated to the gauge field $\vec{A}^{em}$. In other respects, this geometry allows an $SO(3)$ symmetry given by usual *linear* rotations of the physical space coordinates, $\vec{x} \to \vec{x}\,' = \mathfrak{R}\,\vec{x}$ with $\mathfrak{R} \in SO(3)$, and the special non-linear transformations of the fourth coordinate,

$$\mathfrak{R}: \quad x^4 \to x'^{\,4} = x^4 + h(\mathfrak{R}, \vec{x})\,, \tag{119}$$

produced by a function h depending on $\mathfrak{R}$ and $\vec{x}$ which must satisfy

$$h(\mathfrak{R}'\mathfrak{R}, \vec{x}) = h(\mathfrak{R}', \mathfrak{R}\vec{x}) + h(\mathfrak{R}, \vec{x})\,, \quad h(1_3, \vec{x}) = 0\,, \tag{120}$$

where here 1_3 is the identity of $SO(3)$. Obviously, this condition guarantees that Eq. (119) defines a representation of the $SO(3)$ group. These transformations preserve the general form of the line element (112) if $\vec{A}^{em}$ transforms *manifestly* covariant under rotations as a vector field, up to a gauge transformation, V being a scalar. In this way one obtains a representation of the group $SO(3) \otimes U(1)_4$ whose transformations,

$$\begin{aligned} & & \vec{x} &\to \vec{x}\,' = \mathfrak{R}\,\vec{x} & (121)\\ \left[\mathfrak{R}, a^4(\vec{x})\right]: & & x^4 &\to x'^{\,4} = x^4 + h(\mathfrak{R}, \vec{x}) + a^4(\vec{x}) & (122)\\ & & \vec{A}^{em}(\vec{x}) &\to \vec{A}'^{\,em}(\vec{x}\,') = \mathfrak{R}\left\{\vec{A}^{em}(\vec{x}) - \vec{\partial}\left[h(\mathfrak{R}, \vec{x}) + a^4(\vec{x})\right]\right\}, & (123)\end{aligned}$$

produced by any $\mathfrak{R} \in SO(3)$ and real function $a^4(\vec{x})$, *combines* isometries and gauge transformations. Hereby we can separate the isometries requiring that for point-independent parameters a^4 the components of the gauge field remain unchanged, i.e. $A_i'^{\,em} = A_i^{em}$. According to Eq. (123), this condition can be written as

$$\vec{\partial}\, h(\mathfrak{R}, \vec{x}) = \vec{A}^{em}(\vec{x}) - \mathfrak{R}^{-1}\vec{A}^{em}(\mathfrak{R}\vec{x})\,, \tag{124}$$

defining the *specific* function h corresponding to the gauge field $\vec{A}^{em}$.

Remark 3 *The isometry transformations of the Euclidean Taub-NUT space, $x \to x' = \phi_{\mathfrak{R},a^4}(x)$ are three-dimensional rotations and x^4 translations that transform $x = (\vec{x}, x^4)$ into $x' = (\vec{x}\,', x'^4)$ according to Eqs. (121) and (122) restricted to point-independent values of a^4, while the function h is defined by Eq. (124).*

These transformations form the isometry group $I(M) = SO(3) \otimes U(1)_4$ of the Euclidean Taub-NUT space M, the universal covering group of which is the external symmetry group $S(M) = SU(2) \otimes U(1)_4$. What is remarkable here is that the representation of $I(M)$ carried by M mixes up linear transformations with non-linear ones involving the function h.

The study of this type of representation is important since it governs the transformation laws of the vectors and tensors under isometries that are the starting points in deriving conserved quantities through the Noether theorem. However, the properties of the isometries will be better understood if we know the analytical expression of the function h. This may be found combining the integration of the equations (124) with some algebraic properties resulted from the condition (120). The main point is to show that the transformation rule (119) of the fourth coordinate of M is given by a representation of the isometry group induced by one of its subgroups. We recall that the $SO(3)$ subgroup of $I(M)$ has three independent one-parameter subgroups, $SO_i(2)$, $i = 1, 2, 3$, each one including rotations $\mathfrak{R}_i(\alpha)$, of angles $\alpha \in [0, 2\pi)$ around the axis i. With this notation any rotation $\mathfrak{R} \in SO(3)$ in the usual Euler parametrization reads $\mathfrak{R}(\alpha, \beta, \gamma) = \mathfrak{R}_3(\alpha)\mathfrak{R}_2(\beta)\mathfrak{R}_3(\gamma)$.

We start with the observation that the special form of the gauge field (113) does not depend on x^4 and has a special form such that all the rotations of the subgroup $SO_3(2)$ satisfy

$$\mathfrak{R}_3 \vec{A}^{em}(\vec{x}) = \vec{A}^{em}(\mathfrak{R}_3 \vec{x}), \quad \forall\, \mathfrak{R}_3 \in SO_3(2). \tag{125}$$

In these conditions we adopt

Definition 16 *The subgroup $H(M) = SO_3(2) \otimes U(1)_4 \subset I(M)$ is the* little group *associated to $\vec{A}^{em}$.*

In what follows we are interested to exploit the existence of the little group focusing on the rotations $\mathfrak{R}_3 \in SO_3(2)$. According to Eqs. (124) and (125) it results that

$$h(\mathfrak{R}_3, \vec{x}) \equiv \hat{h}(\mathfrak{R}_3) \tag{126}$$

is point-independent being defined only on $SO_3(2)$. Then the condition (120) becomes

$$\hat{h}(\mathfrak{R}_3 \mathfrak{R}'_3) = \hat{h}(\mathfrak{R}_3) + \hat{h}(\mathfrak{R}'_3), \quad \forall\, \mathfrak{R}_3, \mathfrak{R}'_3 \in SO_3(2), \tag{127}$$

which means that the set $\{\hat{h}(\mathfrak{R}_3)\,|\,\mathfrak{R}_3 \in SO_3(2)\}$ forms a one-dimensional representation of the $SO_3(2)$ group provided $\hat{h}(1_3) = 0$. This representation is non-trivial (with $\hat{h}(\mathfrak{R}_3) \neq 0$ when $\mathfrak{R}_3 \neq 1_3$) only if we assume that

$$\hat{h}[\mathfrak{R}_3(\alpha)] = \text{const.}\, \alpha\,. \tag{128}$$

These properties suggests us to write the function h using rotations in the Euler parametrization and the chart with spherical coordinates where the differential equations could be simpler since $h(\mathfrak{R}, \vec{x}) = h(\mathfrak{R}, \theta, \varphi)$ does not depend on the radial coordinate r.

Theorem 21 *In the chart with spherical coordinates the solution the system (124) with the condition (128) reads*

$$\begin{aligned} h[\mathfrak{R}(\alpha,\beta,\gamma),\theta,\varphi] &= -\mu(\alpha+\gamma) \\ &\quad -2\mu \arctan\left[\frac{\sin(\varphi+\gamma)}{\cot\frac{\theta}{2}\cot\frac{\beta}{2} - \cos(\varphi+\gamma)}\right], \end{aligned} \tag{129}$$

for any $\mathfrak{R} \in SO(3)$.

Proof: According to Eqs. (126) and (127), we can write

$$\begin{aligned} h[\mathfrak{R}(\alpha,\beta,\gamma),\vec{x}] &= \hat{h}[\mathfrak{R}_3(\alpha)] + h[\mathfrak{R}_2(\beta)\mathfrak{R}_3(\gamma),\vec{x}] \\ &= h[\mathfrak{R}_2(\beta),\mathfrak{R}_3(\gamma)\vec{x}] + \hat{h}[\mathfrak{R}_3(\alpha+\gamma)], \end{aligned} \tag{130}$$

pointing out that the central problem is to integrate the system (124) in spherical coordinates for the particular case of $\mathfrak{R} = \mathfrak{R}_2(\beta)$. Denoting $h[\mathfrak{R}_2(\beta), \vec{x}] \equiv h(\beta, \theta, \varphi)$, after a few manipulation we find that Eqs. (124) are equivalent with

$$\partial_\theta h(\beta,\theta,\varphi) = -\mu \frac{\sin\varphi \sin\beta}{1+\cos\theta\cos\beta - \sin\theta\cos\varphi\sin\beta}, \tag{131}$$

$$\partial_\varphi h(\beta,\theta,\varphi) = \mu \frac{(1-\cos\theta)(1-\cos\beta) - \sin\theta\cos\varphi\sin\beta}{1+\cos\theta\cos\beta - \sin\theta\cos\varphi\sin\beta}. \tag{132}$$

The integration of this system gives $h(\beta, \theta, \psi)$ up to some arbitrary integration constants resulting from Eq. (128) with *const.* $= -\mu$, as it is shown in Ref. [20]. ∎

The last step here is to show that the function $h(\mathfrak{R}, \theta, \varphi)$ can be easily found by using the technique of induced representations in the chart with spherical coordinates $(r, \theta, \varphi, \chi)$ where θ and φ are the Euler angles of the rotation giving $\vec{x} = \mathfrak{R}(\theta, \varphi, 0)\vec{x}_o$ from $\vec{x}_o = (0, 0, r)$. After an arbitrary rotation $\mathfrak{R}(\alpha, \beta, \gamma) \in I(M)$ we arrive to the chart with the new coordinates $(r, \theta', \varphi', \chi')$ among them the first three are the spherical coordinates of the transformed vector

$$\begin{aligned} \vec{x}\,' &= \mathfrak{R}(\varphi',\theta',0)\vec{x}_o = \mathfrak{R}(\alpha,\beta,\gamma)\vec{x} \\ &= \mathfrak{R}(\alpha,\beta,\gamma)\mathfrak{R}(\varphi,\theta,0)\vec{x}_o\,. \end{aligned} \tag{133}$$

In this context the previous theorem allows us to understand the meaning of the transformation rule of the fourth spherical coordinate [20].

Corollary 10 *The spherical coordinate χ transforms under rotations according to a representation* induced *by the natural representation of the group $SO_3(2) \subset H(M)$ such that the transformed spherical coordinates satisfy*

$$\mathfrak{R}(\varphi', \theta', \chi') = \mathfrak{R}(\alpha, \beta, \gamma)\mathfrak{R}(\varphi, \theta, \chi)\,, \tag{134}$$

for any $\mathfrak{R}(\alpha, \beta, \gamma) \in SO(3)$.

Proof: If we assume that this is true we find the transformation rule of the induced representation

$$\begin{aligned}\mathfrak{R}_3(\chi' - \chi) &= \mathfrak{R}^{-1}(\varphi', \theta', 0)\mathfrak{R}(\alpha, \beta, \gamma)\mathfrak{R}(\varphi, \theta, 0)\\ &= \mathfrak{R}^{-1}(\varphi' - \alpha, \theta', 0)\mathfrak{R}_2(\beta)\mathfrak{R}(\varphi + \gamma, \theta, 0)\,.\end{aligned} \tag{135}$$

Furthermore, we express this equation in terms of $SU(2)$ transformations corresponding to all the particular rotations involved therein and calculate the transformed coordinates and

$$h(\mathfrak{R}, \theta, \varphi) = -\mu(\chi' + \varphi' - \chi - \varphi)\,, \tag{136}$$

that is just the function h given by Theorem 21. ■
Thus we have shown that the transformation law of the fourth spherical coordinate is given by the induced representation (135).

4.2 The angular momentum and related operators

In the quantum theory on the Euclidean Taub-NUT background the basic operators are introduced using the geometric quantization. Now when we know the closed form of the function h we can calculate directly the components of the Killing vectors and the generators of the natural representation of the isometry group using only group theoretical methods.

The generators of the natural representation of $I(M)$ are the (orbital) differential operators (42). They can be calculated starting with a set of parameters ξ^a and the functions $\phi_{\mathfrak{R},a^4} \equiv \phi_\xi(x) = x'(x, \xi)$ that give the Killing vectors k_a according to Eq. (40). In the case of our isometry group we take the first three parameters, ξ^i, the Cayley-Klein parameters of the rotations $\mathfrak{R}(\vec{\xi}) \in SO(3)$ and we denote $\xi^4 = a^4$. Then we find that the generator of the $U(1)_4$ translations is the fourth component of the momentum operator, $P_4 = -i\partial_4$ since $k^i_{(4)} = 0$ and $k^4_{(4)} = 1$.

Theorem 22 *The $SO(3)$ generators of the natural representation are the components of the orbital angular momentum operator:*

$$\begin{aligned}L_1 &= -i(x^2\partial_3 - x^3\partial_2) + i\mu\,\frac{x^1}{r + x^3}\partial_4\,,\\ L_2 &= -i(x^3\partial_1 - x^1\partial_3) + i\mu\,\frac{x^2}{r + x^3}\partial_4\,,\\ L_3 &= -i(x^1\partial_2 - x^2\partial_1) + i\mu\,\partial_4\,.\end{aligned} \tag{137}$$

Proof: The first terms of the angular momentum correspond to the usual linear transformation $\vec{x}\,' = \Re\,\vec{x}$ giving the components $k^i_{(j)} = \varepsilon_{ijk}x^k$ of the first three Killing vectors in the basis of the Cartesian natural frame. The contributions due to h have to be calculated according to Eqs. (40) and (119) starting with

$$\frac{\partial}{\partial\beta}h(\beta,\theta,\phi)\Big|_{\beta=0} = -\mu\frac{\sin\theta\sin\phi}{1+\cos\theta}\,. \tag{138}$$

Then, denoting $\xi^2=\beta$ for $\alpha=\gamma=0$ and $\xi^3=\alpha$ for $\beta=\gamma=0$ and using a simple rotation of angle $\pi/2$ around the third axis we find

$$k^4_{(1,2)} = \frac{\partial h(\Re,\vec{x})}{\partial\xi^{1,2}}\Big|_{\xi=0} = -\mu\frac{x^{1,2}}{r+x^3}\,, \quad k^4_{(3)} = \frac{\partial h(\Re,\vec{x})}{\partial\xi^3}\Big|_{\xi=0} = -\mu\,. \tag{139}$$

Finally from Eq. (42) we obtain the operators (137) corresponding to the Caley-Klein parameters ξ^i. ■

In the Cartesian charts one can choose a diagonal gauge suitable for physical interpretation. This is given by the gauge fields $\hat{e}^{\hat\alpha}$ and $e_{\hat\alpha}$ having the non-vanishing components [51]

$$\begin{aligned}
\hat{e}^i_j &= \frac{1}{\sqrt{V}}\delta_{ij}\,, \quad \hat{e}^4_i = \sqrt{V}A^{em}_i\,, \quad \hat{e}^4_4 = \sqrt{V}\,,\\
e^i_j &= \sqrt{V}\delta_{ij}\,, \quad e^4_i = -\sqrt{V}A^{em}_i\,, \quad e^4_4 = \frac{1}{\sqrt{V}}\,,
\end{aligned} \tag{140}$$

in the natural Cartesian frame. This gauge fixing defines local orthogonal frames where the Killing vectors $k_{(i)}$ yield by the previous theorem have the components

$$k^i_{(j)} = \frac{1}{\sqrt{V}}\varepsilon_{ijk}x^k\,, \quad k^4_{(i)} = -\mu\frac{x^i}{r}\sqrt{V}\,, \tag{141}$$

which covariantly transform under linear $SO(3)$ rotations. This behavior is rather surprising in view of the fact that the fourth coordinate transforms under rotations according to the induced representation (122). The explanation of this remarkable phenomenon is given by

Theorem 23 *In the local frames of the Euclidean Taub-NUT space defined by the gauge (140) the representation vect$[S(M)]$ of the group $S(M)=SU(2)\otimes U(1)_4$ is just the* linear fundamental *representation of the group $I(M)=SO(3)\otimes U(1)_4$.*

Proof: The representation of the $U(1)_4$ translations is anyway the usual one such that the problem here is to calculate the behavior under rotations. Starting with the isometries $\phi_{\vec\xi}=\phi_{\Re(\vec\xi),a^4=0}$ and $(A_{\vec\xi},\,\phi_{\vec\xi})\in SU(2)\subset S(M)$ and using Eqs. (124) and (140), we find that the matrices defined by Eq. (44) are point-independent having the non-vanishing matrix elements $\Lambda^{i\cdot}_{\cdot j}[A_{\vec\xi}(x)]=\Re_{ij}(\vec\xi)$ and $\Lambda^{4\cdot}_{\cdot 4}[A_{\vec\xi}(x)]=1$. Thus

we see that in the local frames given by this gauge the fourth component of any vector behaves as a scalar under rotations. ■
We specify that here it is crucial to consider the group $S(M)$ instead of $I(M)$ since only the transformations of the external symmetry group preserve this gauge showing off the $SO(3)$ symmetry as a *global* one.

In this context one can correctly define the three-dimensional physical momentum $\vec{P}$ whose components in the above defined local frames are

$$P_i = -i\frac{1}{\sqrt{V}}e_i^{\mu}\partial_{\mu} = -i(\partial_i - A_i^{em}\partial_4)\,, \tag{142}$$

obeying $[P_i, P_j] = i\varepsilon_{ijk}B_k^{em}P_4$, $[P_i, P_4] = 0$ and $[L_i,\, P_j] = i\varepsilon_{ijk}P_k$ which indicate that $\vec{P}$ behaves as a vector under rotations. In addition, the angular momentum can be written in covariant form as

$$\vec{L} \,=\, \vec{x}\times\vec{P} - \mu\frac{\vec{x}}{r}P_4\,. \tag{143}$$

The scalar quantum mechanics in the Taub-NUT geometry [48] is based on the Schrödinger or Klein-Gordon equations involving the static operator

$$\Delta = -\nabla_{\mu}g^{\mu\nu}\nabla_{\nu} = V\vec{P}^{\,2} + \frac{1}{V}{P_4}^2\,, \tag{144}$$

which is either proportional with the Hamiltonian operator of the Schrödinger theory or represents the static part of the Klein-Gordon operator [18]. In both cases we are interested to find operators commuting with Δ since these give rise to the conserved quantities with physical significance.

The Euclidean Taub-NUT space is a hyper-Kähler manifold possessing a triplet of real unit roots (i.e., a hypercomplex structure), $\boldsymbol{f} = \{f^{(1)},\, f^{(2)},\, f^{(3)}\}$, defined as

$$f^{(i)} = f^{(i)}_{\hat{\alpha}\hat{\beta}}\hat{e}^{\hat{\alpha}}\wedge\hat{e}^{\hat{\beta}} = 2\hat{e}^i\wedge\hat{e}^4 - \varepsilon_{ijk}\hat{e}^j\wedge\hat{e}^k\,. \tag{145}$$

In addition, there exists a fourth K-Y tensor,

$$f^Y = f^Y_{\hat{\alpha}\hat{\beta}}\hat{e}^{\hat{\alpha}}\wedge\hat{e}^{\hat{\beta}} = \frac{x^i}{r}f^{(i)} + \frac{2x^i}{\mu V}\varepsilon_{ijk}\,\hat{e}^j\wedge\,\hat{e}^k\,, \tag{146}$$

which is not covariantly constant. The presence of f^Y is due to the existence of the hidden symmetries of the Euclidean Taub-NUT geometry which are encapsulated in three non-trivial S-K tensors and interpreted as the components of the so-called Runge-Lenz vector of the Euclidean Taub-NUT problem. These S-K tensors can be expressed as symmetrized products of K-Y tensors [14, 13],

$$k_{(i)\mu\nu} = \frac{\mu}{4}(f^Y_{\ \mu\lambda}f^{(i)\lambda}_{\ \ \ \nu} + f^Y_{\ \nu\lambda}f^{(i)\lambda}_{\ \ \ \mu}) + \frac{1}{2\mu}(k_{(4)\mu}k_{(i)\nu} + k_{(4)\nu}k_{(i)\mu})\,, \tag{147}$$

and with their help one defines the vector operator

$$\vec{K} = -\frac{1}{2}\nabla_\mu \vec{k}^{\mu\nu}\nabla_\nu = \frac{1}{2}\left(\vec{P}\times\vec{L}-\vec{L}\times\vec{P}\right) - \mu\frac{\vec{x}}{r}\left(\frac{1}{2}\Delta - P_4^2\right), \tag{148}$$

which play the same role as the Runge-Lenz vector operator in the usual quantum mechanical Kepler problem [48]. This transforms as a vector under the rotations of $vect[S(M)]$ such that one can write the following complete system of commutation relations

$$\begin{aligned}
[L_i,\, L_j] &= i\varepsilon_{ijk}\, L_k\,,\\
[L_i,\, K_j] &= i\varepsilon_{ijk}\, K_k\,,\\
[K_i,\, K_j] &= i\varepsilon_{ijk}\, L_k B^2\,,
\end{aligned} \tag{149}$$

where $B^2 = {P_4}^2 - \Delta$. The operators L_i and K_i commute with B since they commute with Δ and P_4. In addition, we observe that the new operators

$$C_1 = \vec{L}^2 B^2 + \vec{K}^2 = \frac{\mu^2}{4}\left(B^2 + {P_4}^2\right)^2 - B^2\,, \quad C_2 = \vec{L}\cdot\vec{K} = 0\,, \tag{150}$$

play the role of Casimir operators for the open algebra (149). We recall that K^i commute with Δ grace to the factorization (147) which eliminates the quantum anomaly.

When one goes to the chart with spherical coordinate a special attention must be paid to the meaning of Eq. (115) which shows that the fourth spherical coordinate χ is *translated* with the angular coordinate φ. This translation is rather unusual being performed by the unitary operator

$$U(\varphi) = e^{i\varphi P_\chi}\,, \tag{151}$$

where $P_\chi = -\mu P_4 = -i\partial_\chi$ replaces the Cartesian operator P_4. Since the differential and local operators are defined often in the coordinate representation of a given chart, we must take care when we change the chart.

Remark 4 *The coordinate representation of the Cartesian chart must be transformed into the* equivalent *coordinate representation of the spherical chart transforming each operator X defined in the Cartesian chart into the equivalent operator*

$$X^{sph} = U(\varphi) X U^\dagger(\varphi) \tag{152}$$

of the spherical chart.

Thus, for example, the components of the orbital angular momentum (137) in the spherical chart and canonical basis (with $L_\pm = L_1 \pm iL_2$) become

$$L_3^{sph} = -i\partial_\varphi\,, \tag{153}$$

$$L_\pm^{sph} = e^{\pm i\varphi}\left[\pm\,\partial_\theta + i\left(\cot\theta\,\partial_\varphi - \frac{1}{\sin\theta}\,\partial_\chi\right)\right]. \tag{154}$$

Many other operators including the Runge-Lenz vector will take new forms in the representation of the spherical coordinates but preserving their commutation relations. However, in current calculations when we do not work simultaneously with both these representations of the operator algebra we drop out the superscript above, denoting the equivalent operators with the same symbol.

4.3 Scalar quantum modes and dynamical algebras

The special form of the $SO(3)$ generators (153) and (154) leads to new spherical harmonics that permit to separate the spherical variables in the static Klein-Gordon equation, $\Delta\mathfrak{U}_E = E^2\mathfrak{U}_E$, giving the eigenfunctions of the operator Δ which represents the squared Hamiltonian operator of the relativistic theory of the scalar field without the explicit mass term.

Definition 17 *The* central regular *modes of the scalar field on M are given by the eigenfunctions of the complete set of commuting operators $\{\Delta,\, P_\chi,\, \vec{L}^{\,2},\, L_3\}$.*

The corresponding eigenvalues E^2, q, $l(l+1)$ and m determine the eigenfunctions

$$\mathfrak{U}^q_{E,l,m}(r,\varphi,\theta,\chi) = \frac{1}{r}\mathfrak{f}_{E,q,l}(r) Y^q_{l,m}(\theta,\varphi,\chi)\,, \tag{155}$$

which have separated variables.

Definition 18 *We call $SO(3)\otimes U(1)$ spherical harmonics the functions $Y^q_{l,m}$ defined on the compact domain $S^2\times D_\chi$, which satisfy the eigenvalue problems*

$$\vec{L}^{\,2} Y^q_{l,m} = l(l+1)\, Y^q_{l,m}\,, \tag{156}$$

$$L_3 Y^q_{l,m} = m\, Y^q_{l,m}\,, \tag{157}$$

$$P_\chi Y^q_{l,m} = q\, Y^q_{l,m}\,, \tag{158}$$

and the orthonormalization condition

$$\left\langle Y^q_{l,m}, Y^{q'}_{l',m'}\right\rangle = \int_{S^2} d(\cos\theta) d\varphi \int_0^{4\pi} d\chi\, Y^q_{l,m}(\theta,\varphi,\chi)^*\, Y^{q'}_{l',m'}(\theta,\varphi,\chi) = \delta_{l,l'}\delta_{m,m'}\delta_{q,q'}\,. \tag{159}$$

These functions form a basis of the Hilbert space of square integrable functions on $S^2\times D_\chi$ since the set of commuting operators $\{\vec{L}^{\,2}, L_3, P_\chi\}$ is complete in this space.

In Ref. [19] we pointed out that the $SO(3)\otimes U(1)$-harmonics, are new spherical harmonics. The usual boundary conditions on $S^2\times D_\chi$ require l and m to be integer numbers and $q = 0, \pm 1/2, \pm 1, ...$ [48] but, in general, q can be any real number. Solving Eqs. (157) and (158) we get

$$Y^q_{l,m}(\theta,\varphi,\chi) = \frac{1}{4\pi}\,\Theta^q_{l,m}(\cos\theta) e^{im\varphi} e^{iq\chi}\,, \tag{160}$$

where the function $\Theta^q_{l,m}$ must satisfy Eq. (156) and the normalization condition

$$\int_{-1}^{1} d(\cos\theta) \left|\Theta^q_{l,m}(\cos\theta)\right|^2 = 2\,, \tag{161}$$

resulted from Eq. (159). This problem has solutions for all the values of the quantum numbers obeying $|q| - 1 < |m| \le l$ when one founds [19]

$$\begin{aligned} \Theta^q_{l,m}(\cos\theta) &= \frac{\sqrt{2l+1}}{2^{|m|}}\left[\frac{(l-|m|)!\,(l+|m|)!}{\Gamma(l-q+1)\Gamma(l+q+1)}\right]^{\frac{1}{2}} \\ &\quad \times (1-\cos\theta)^{\frac{|m|-q}{2}}\,(1+\cos\theta)^{\frac{|m|+q}{2}}\; P^{(|m|-q,\,|m|+q)}_{l-|m|}(\cos\theta)\,. \end{aligned} \tag{162}$$

For $m = |m|$ the $SO(3) \otimes U(1)$ harmonics are given by (160) and (162) while for $m < 0$ we have to use the obvious formula

$$Y^q_{l,-m} = (-1)^m \left(Y^{-q}_{l,m}\right)^*\,. \tag{163}$$

When the boundary conditions allow half-integer quantum numbers l and m then we say that the functions defined by Eqs. (160) and (162) (up to a suitable factor) represent $SU(2) \otimes U(1)$ harmonics. Thus we have obtained a non-trivial generalization of the spherical harmonics of the same kind as the spin-weighted spherical harmonics [52] or those studied in [53]. Indeed, if l, m and $q = m'$ are either integer or half-integer numbers then we have

$$Y^{m'}_{l,m}(\theta,\varphi,\chi) = \frac{\sqrt{2l+1}}{4\pi} D^l_{m,m'}(\varphi,\theta,\chi)\,, \tag{164}$$

where $D^l_{m,m'}$ are the matrix elements of the irreducible representation of weight l of the $SU(2)$ group corresponding to the rotation of Euler angles (φ,θ,χ). What is new here is that our harmonics are defined for any real number q. For this reason these are useful in solving some actual physical problems [54]. Notice that similar spherical harmonics were used recently in [55] under the name of ring-shaped harmonics.

Turning back to the scalar modes on M and using the identity

$$\vec{P}^2 = -\partial_r^2 - \frac{2}{r}\partial_r + \frac{1}{r^2}\vec{L}^2 - \frac{1}{r^2}P_\chi^2\,, \tag{165}$$

we find that the radial wave functions $\mathfrak{f}$ satisfy the radial equation

$$\left[-\frac{d^2}{dr^2} + \frac{l(l+1)}{r^2} - \frac{a}{r}\right]\mathfrak{f}_{E,q,l}(r) = -b^2\,\mathfrak{f}_{E,q,l}(r)\,, \tag{166}$$

whose parameters $a = \mu\left[E^2 - 2\hat{q}^2\right]$ and $b^2 = \hat{q}^2 - E^2$ depend on the eigenvalues of P_4 denoted by $\hat{q} = -q/\mu$. Notice that here b^2 is the eigenvalue of the operator B^2

for given E and q. Looking for the particular solutions of the radial equation on the non-compact domain D_r we have to select either square integrable functions with respect to the radial scalar product [19]

$$\langle \mathfrak{f}_{E,q,l}, \mathfrak{f}_{E',q,l}\rangle = \int_{D_r} dr \left|1+\frac{\mu}{r}\right| \mathfrak{f}_{E,q,l}(r)^* \mathfrak{f}_{E',q,l}(r)\,, \tag{167}$$

or solutions that behave as tempered distributions on D_r. One obtains thus a Kepler-like problem similar to the well-known one of the non-relativistic quantum mechanics, the only differences being the parametrization and the form of the scalar product. The particular solution of Eq. (166) can be written in terms of the confluent hypergeometric function as

$$\mathfrak{f}_{E,q,l}(r) = N_{E,q,l}\, r^{l+1} e^{-2br} F(s, 2l+2, 2br)\,, \quad s = l+1-\frac{a}{2b}\,, \tag{168}$$

where $N_{E,q,l}$ is the normalization constant. It is easy to show that for $\mu > 0$ the radial wave functions are not square integrable and, therefore, the energy spectrum is continuous in the domain $E \ge |\hat{q}|$. The case of $\mu < 0$ is most interesting since beside the mentioned continuous energy spectrum there is a discrete spectrum for $a > 0$ and $b^2 > 0$ when $0 < E < |\hat{q}|$. Indeed, then the quantization condition $s = -n_r$, $n_r = 0, 1, 2, ...$ gives the usual formula $a = 2n|b|$ where $n = n_r + l + 1$ is the *principal* quantum number. Hereby one finds the energy levels

$${E_n}^2 = \frac{2}{\mu^2}\left[n\sqrt{n^2-q^2} - (n^2-q^2)\right]\,, \tag{169}$$

for all $n > |q| > 0$ [48]. This spectrum is countable and finite since $\lim_{n\to\infty} E_n = \hat{q}$.

Remark 5 *The condition $E > 0$ guarantees that there are no zero modes and, therefore, the operator Δ is invertible.*

Another problem is to define the generators of the dynamical algebras corresponding to different spectral domains. This can be done as in the case of the standard quantum Kepler problem *rescaling* the operators K_i in order to close up the commutation relations (149). For given values of E and $\hat{q}$ the rescaled operators

$$K_i^{re} = \left\{ \begin{array}{lllll} B^{-1}K_i & \text{for} & \mu<0 & \text{and} & E<|\hat{q}| \\ K_i & \text{for} & \text{any } \mu & \text{and} & E=|\hat{q}| \\ \pm iB^{-1}K_i & \text{for} & \text{any } \mu & \text{and} & E>|\hat{q}| \end{array}\right. \tag{170}$$

and L_i $(i = 1, 2, 3)$ generate either a representation of the $o(4)$ algebra for $\mu < 0$ and discrete energy spectrum or a representation of the $o(3, 1)$ algebra for the continuous spectrum in the domain $E > |\hat{q}|$. A special case is that of the dynamical algebra $e(3)$ which corresponds only to the ground energy of the continuous spectrum, $E = |\hat{q}|$.

In general, the dynamical algebras may help us to analyze the structure of the spaces of the eigenvectors of Δ seen as carrier spaces of several irreducible representations of the dynamical algebra. For example, in the case of the discrete energy spectrum the $so(4)$ dynamical algebra generated by L_i and $K_i^{re} = K_i B^{-1}$ has the Casimir operators $C_1^{re} = C_1 B^{-2} = \vec{L}^2 + (\vec{K}^{re})^2$ allowing the eigenvalues $c_1 = n^2 - 1$ (with $n > |q|$) and $C_2^{re} = C_2 B^{-1} = \vec{L} \cdot \vec{K}^{re} = 0$, as it results from Eqs. (150). Thus the eigenspace of the energy level E_n appears as the carrier space of the unitary and irreducible representation of $so(4)$ having the $su(2)$ weights $(\frac{n-1}{2}, \frac{n-1}{2})$. However, for the dynamical algebras $so(3,1)$ or $e(3)$ the problem is more complicated since their unitary representations are no longer finite-dimensional.

Finally we note that there are other interesting scalar modes that do not present central symmetry. Thus we have show that the complete set of commuting operators $\{\Delta, P_\chi, K_3, L_3\}$ gives *axial* modes that can be completely solved in parabolic coordinates [19]. The discrete axial modes are determined by the set of eigenvalues ${E_n}^2, q, \kappa$ and m that depend on the integer numbers n_1, $n_2 = 0, 1, 2, ...$ and m (obeying $|m| > |q| - 1$) which give the principal quantum number $n = n_1 + n_2 + |m| + 1$ and the eigenvalue $\kappa = |b|(n_2 - n_1 - q)$ of the operator K_3.

4.4 Conserved Dirac and Pauli operators

For building the Dirac theory we consider the Cartesian chart, the usual four-dimensional space of the Dirac spinors, Ψ, and the Dirac matrices $\gamma^{\hat\alpha}$, that satisfy $\{\gamma^{\hat\alpha}, \gamma^{\hat\beta}\} = 2\delta_{\hat\alpha\hat\beta}$, in the following representation

$$\gamma^i = -i \left(\begin{array}{cc} 0 & \sigma_i \\ -\sigma_i & 0 \end{array} \right), \quad \gamma^4 = \left(\begin{array}{cc} 0 & \mathbf{1}_2 \\ \mathbf{1}_2 & 0 \end{array} \right), \tag{171}$$

where σ_i are the Pauli matrices. In addition we take $\gamma = \mathbf{1}$ and denote by $\gamma^5 = \gamma^1\gamma^2\gamma^3\gamma^4 = \mathrm{diag}(\mathbf{1}_2, -\mathbf{1}_2)$ the chiral matrix ($\gamma^5 = \gamma^{ch} = F$) which is just the matrix γ^0 of the Kaluza-Klein theory explicitly depending on time [8]. In this representation adopted here all the gamma-matrices are self-adjoint with respect to the Dirac adjoint ($\overline{X} = X^+$) and the Euclidean metric is of positive signature for a pure space-like manifold. Therefore, it is convenient to change some phase factors of the operators we define here as indicated in Remark 1.

Let us start with the *standard* Dirac operator without explicit mass term defined now as $D = \gamma^\alpha \nabla_\alpha$ [8, 15]. This is related to the Hamiltonian operator [8, 16]

$$H = \gamma^5 D = \left(\begin{array}{cc} 0 & \boldsymbol{\alpha}^* \\ \boldsymbol{\alpha} & 0 \end{array} \right) = \left(\begin{array}{cc} 0 & V\pi^* \frac{1}{\sqrt{V}} \\ \sqrt{V}\pi & 0 \end{array} \right), \tag{172}$$

that, after a little calculation, can be expressed in terms of $\pi = \sigma_P - iV^{-1}P_4$ and $\pi^* = \sigma_P + iV^{-1}P_4$ depending on $\sigma_P = \vec{\sigma} \cdot \vec{P}$. These operators obey

$$\Delta = \boldsymbol{\alpha}^* \boldsymbol{\alpha} = V\pi^*\pi\,. \tag{173}$$

We specify that here the star superscript is a mere notation that does not coincide with the Hermitian conjugation at the level of the Pauli operators which enter in the structure of the basic Dirac operators. The Hamiltonian operator that is the central piece of the Dirac theory has remarkable properties.

Theorem 24 *The spectrum of the Hamiltonian operator coincides with the energy spectrum of the operator Δ.*

Proof: From Eqs. (172) and (173) we see that the eigenvalue problem $H\psi_E = E\psi_E$ is solved by the Dirac spinors $\psi_E = (u_E,\, E^{-1}\boldsymbol{\alpha} u_E)^T$ if the first Pauli spinor u_E satisfies $\Delta u_E = E^2 u_E$. Consequently, E is an eigenvalue of H only if E^2 is an eigenvalue of the operator Δ. ■

The main consequence is that the operator H is *invertible* since there are no zero modes.

The operators we intend to study here are operators of the Dirac theory which *commute* with the Hamiltonian operator (172).

Definition 19 *We say that the Dirac operators which commute with H are conserved.*

We denote by $\boldsymbol{D} = \{X \,|\, [X, H] = 0\}$ the algebra of the conserved Dirac operators observing that they can be related to Pauli operators commuting with Δ which form the algebra $\boldsymbol{P} = \{\hat{X} \,|\, [\hat{X}, \Delta] = 0\}$ where we include the orbital operators having this property. All these operators are considered as conserved operators in the sense of the Klein-Gordon theory. Notice that the Pauli operators are interesting here since they are involved in different versions of the dyon theory [56] which may be compared to our approach.

Theorem 25 *The Pauli blocks, $\hat{X}^{(ab)}$ $(a, b = 1, 2)$, of any operator*

$$X = \left(\begin{array}{cc} \hat{X}^{(11)} & \hat{X}^{(12)} \\ \hat{X}^{(21)} & \hat{X}^{(22)} \end{array} \right) \in \boldsymbol{D}\,, \tag{174}$$

must satisfy the conditions $\hat{X}^{(21)} = \boldsymbol{\alpha}\hat{X}^{(12)}\boldsymbol{\alpha}\Delta^{-1}$ and $\hat{X}^{(11)}$, $\hat{X}^{(12)}\boldsymbol{\alpha}$, $\boldsymbol{\alpha}^*\hat{X}^{(21)} \in \boldsymbol{P}$.

Proof: From $[X, H] = 0$ it results the equivalent system

$$\hat{X}^{(22)}\boldsymbol{\alpha} = \boldsymbol{\alpha}\hat{X}^{(11)}\,, \quad \boldsymbol{\alpha}^*\hat{X}^{(22)} = \hat{X}^{(11)}\boldsymbol{\alpha}^*\,, \tag{175}$$

$$\hat{X}^{(12)}\boldsymbol{\alpha} = \boldsymbol{\alpha}^*\hat{X}^{(21)}\,, \quad \boldsymbol{\alpha}\hat{X}^{(12)} = \hat{X}^{(21)}\boldsymbol{\alpha}^*\,, \tag{176}$$

giving $\hat{X}^{(21)}$ and $[\hat{X}^{(11)}, \Delta] = [\hat{X}^{(12)}\boldsymbol{\alpha}, \Delta] = [\boldsymbol{\alpha}^*\hat{X}^{(21)}, \Delta] = 0\,.$ ■

We observe that possible solutions of Eqs. (175) and (176) are the diagonal operators

$$\mathcal{D}(\hat{X}) = \left(\begin{array}{cc} \hat{X} & 0 \\ 0 & \boldsymbol{\alpha}\hat{X}\Delta^{-1}\boldsymbol{\alpha}^* \end{array} \right)\,, \tag{177}$$

where $\hat{X} \in \boldsymbol{P}$. Particularly, for $\hat{X} = \mathbf{1}_2$ we obtain the projection operator

$$I = \mathcal{D}(\mathbf{1}_2) = \left(\begin{array}{cc} \mathbf{1}_2 & 0 \\ 0 & \boldsymbol{\alpha}\Delta^{-1}\boldsymbol{\alpha}^* \end{array} \right), \tag{178}$$

on the space $\Psi_D = I\Psi$ in which the eigenspinors ψ_E of H form a (generalized) basis. This projection operator splits the algebra $\boldsymbol{D} = \boldsymbol{D}_0 \oplus \boldsymbol{D}_1$ in two subspaces of the projections $XI \in \boldsymbol{D}_0$ and $X(\mathbf{1} - I) \in \boldsymbol{D}_1$ of all $X \in \boldsymbol{D}$.

Theorem 26 *The subalgebra $\boldsymbol{D}_1$ is an ideal in $\boldsymbol{D}$.*

Proof: According to Eqs. (175) and (176) we find that the projections of two arbitrary operators $X, Y \in \boldsymbol{D}$ satisfy $(XI)(YI) = (XY)I$ and $[X(\mathbf{1} - I)](YI) = 0$ which lead to the conclusion that $\boldsymbol{D}_0$ is a subalgebra while $\boldsymbol{D}_1$ is even an ideal in $\boldsymbol{D}$. Obviously, I is the identity operator of $\boldsymbol{D}_0$. ■

In [8] we introduced the $\mathcal{Q}$-operators defined as

$$\mathcal{Q}(\hat{X}) = \left\{ H\,, \left(\begin{array}{cc} \hat{X} & 0 \\ 0 & 0 \end{array} \right) \right\} = \left(\begin{array}{cc} 0 & \hat{X}\boldsymbol{\alpha}^* \\ \boldsymbol{\alpha}\hat{X} & 0 \end{array} \right), \tag{179}$$

where $\hat{X}$ may be any Pauli operator. However, if $\hat{X} \in \boldsymbol{P}$ then $\mathcal{Q}(\hat{X}) \in \boldsymbol{D}_0$ since $[\mathcal{Q}(\hat{X}), H] = 0$ and $\mathcal{Q}(\hat{X})I = \mathcal{Q}(\hat{X})$. If $\hat{X} = \mathbf{1}_2$ we obtain just the Hamiltonian operator $H = \mathcal{Q}(\mathbf{1}_2) \in \boldsymbol{D}_0$. Consequently, the inverse of H with respect to I can be represented as $H^{-1} = \mathcal{Q}(\Delta^{-1})$. The mappings $\mathcal{D} : \boldsymbol{P} \to \boldsymbol{D}_0$ and $\mathcal{Q} : \boldsymbol{P} \to \boldsymbol{D}_0$ are linear and have the following algebraic properties

$$\mathcal{D}(\hat{X})\mathcal{D}(\hat{Y}) = \mathcal{D}(\hat{X}\hat{Y})\,, \tag{180}$$

$$\mathcal{Q}(\hat{X})\mathcal{Q}(\hat{Y}) = \mathcal{D}(\hat{X}\hat{Y}\Delta)\,, \tag{181}$$

$$\mathcal{D}(\hat{X})\mathcal{Q}(\hat{Y}) = \mathcal{Q}(\hat{X})\mathcal{D}(\hat{Y}) = \mathcal{Q}(\hat{X}\hat{Y})\,, \tag{182}$$

for any $\hat{X}, \hat{Y} \in \boldsymbol{P}$. Moreover, the relations $[\gamma^5, \mathcal{D}(\hat{X})] = 0$ and $\{\gamma^5, \mathcal{Q}(\hat{X})\} = 0$ show us that, according to the usual terminology [57], $\mathcal{D}$ and $\gamma^5\mathcal{D}$ are *even* Dirac operators while $\mathcal{Q}$ and $\gamma^5\mathcal{Q}$ are *odd* ones. We note that there are many other odd or even operators which do not have such forms.

Since I is the projection operator on the space of the Dirac spinors Ψ_D we adopt

Definition 20 *We say that the projection IXI of any Dirac operator X represents the* physical part *of X.*

One can convince ourselves that if $X \in \boldsymbol{D}$ then

$$IXI \equiv XI = \mathcal{D}(\hat{X}^{(11)}) + \mathcal{Q}(\hat{X}^{(12)}\boldsymbol{\alpha}\Delta^{-1})\,, \tag{183}$$

which means that all the operators of $\boldsymbol{D}_0$ can be written in terms of $\mathcal{D}$ or $\mathcal{Q}$-operators. Thus the action of X reduces to that of the Pauli operators involved in (183) allowing us to rewrite the problems of the Dirac theory in terms of Pauli operators [15, 16]. Indeed, it is easy to show that the action of any operator $X \in \boldsymbol{D}$ on $\psi_E \in \Psi_D$ is

$$X\psi_E = XI\psi_E = \left(\begin{array}{c} \hat{\mathcal{P}}_E(X)\, u_E \\ E^{-1}\boldsymbol{\alpha}\hat{\mathcal{P}}_E(X)\, u_E \end{array} \right), \tag{184}$$

where, by definition,

$$\hat{\mathcal{P}}_E(X) = \hat{X}^{(11)} + E^{-1}\hat{X}^{(12)}\boldsymbol{\alpha} \in \boldsymbol{P} \tag{185}$$

is the Pauli operator *associated* to X. Since the mapping $\hat{\mathcal{P}}_E : \boldsymbol{D} \to \boldsymbol{P}$ is linear and satisfies $\hat{\mathcal{P}}_E(X) = \hat{\mathcal{P}}_E(XI)$ it results that $\mathrm{Ker}\, \hat{\mathcal{P}}_E = \boldsymbol{D}_1$. In other respects, Eqs. (175) and (176) lead to the important property

$$\hat{\mathcal{P}}_E(XY) = \hat{\mathcal{P}}_E(X)\hat{\mathcal{P}}_E(Y)\,, \quad \forall X, Y \in \boldsymbol{D}\,. \tag{186}$$

which guarantees that $\hat{\mathcal{P}}_E$ preserves the algebraic relations, mapping any algebra or superalgebra of $\boldsymbol{D}_0$ into an *isomorphic* algebra or superalgebra of $\boldsymbol{P}$, with the same commutation and anticommutation rules.

The conclusion here is that only the diagonal conserved Dirac operators can be correctly associated to conserved Pauli operators independent on E. However, the off-diagonal operators can be transformed at any time in diagonal ones using the multiplication with H or H^{-1}. For example, H itself which is off-diagonal is related to the diagonal operators $H^2 = \mathcal{D}(\Delta)$ or I. Thus each Dirac operator from $\boldsymbol{D}$ can be brought in a diagonal form associated with an operator from $\boldsymbol{P}$.

4.5 The operators of the Dirac theory

The simplest operators of $\boldsymbol{D}$ which commute with H, D, and γ^5 are the generators (50) of the representation $spin[S(M)]$ carried by the space Ψ. As observed before, the expressions of these operators are strongly dependent on the gauge fixing.

Theorem 27 *In the gauge (140) the spinor fields transform* manifestly covariant *under the transformations of the group $S(M) = SU(2) \otimes U(1)_4$.*

Proof: Using the components of the Killing vectors given by Theorem 22 and calculating the functions (51) we find that in this gauge the rotation generators of $spin[S(M)]$ are the standard components of the *total* angular momentum

$$\mathcal{J}_i = L_i + S_i\,, \quad S_i = \tfrac{1}{2}\varepsilon_{ijk}S^{jk} = \tfrac{1}{2}\mathrm{diag}(\sigma_i, \sigma_i)\,, \tag{187}$$

with point-independent spin operators, [8]. In the same way one can show that the $U(1)_4$ generator, P_4, does not get a spin term. ■

Hence it results that the representation $spin[S(M)]$ is *reducible* being a sum of two irreducible representations carried by spaces of two-dimensional Pauli spinors where the components of the total angular momentum are $J_i = L_i + \frac{1}{2}\sigma_i$. This means that we can put

$$\mathcal{J}_i I = \mathcal{D}(J_i) = \mathcal{D}(L_i) + \tfrac{1}{2}\mathcal{D}(\sigma_i)\,, \tag{188}$$

where both the orbital and the spin terms *separately* commute with H since L_i and σ_i commute with Δ.

The triplet $\boldsymbol{f}$ defined by Eq. (145) gives rise to the spin-like operators

$$\Sigma^{(i)} = \frac{i}{4}\hat{f}^{(i)}_{\hat{\alpha}\hat{\beta}}\gamma^{\hat{\alpha}}\gamma^{\hat{\beta}} = \left(\begin{array}{cc} \sigma_i & 0 \\ 0 & 0 \end{array}\right), \tag{189}$$

and, according to Eq. (76), produce the Dirac-type operators [8]

$$D^{(i)} = -f^{(i)}_{\mu,\nu}\gamma^{\nu}\nabla^{\mu} = i[D,\,\Sigma^{(i)}] = -i\left(\begin{array}{cc} 0 & \sigma_i\boldsymbol{\alpha}^* \\ \boldsymbol{\alpha}\sigma_i & 0 \end{array}\right) = -i\mathcal{Q}(\sigma_i)\,, \tag{190}$$

which anticommute with D and γ^5. The operators D and $D^{(i)}$, $i = 1, 2, 3$, form the basis of the $\mathcal{N} = 4$ superalgebra $\boldsymbol{d}_{\boldsymbol{f}} \subset \boldsymbol{D}_0$ with the same anticommutation relations as Eq. (96). According to the general theory presented in the previous section, the spinor representation the of group $(G_{\boldsymbol{f}}) \sim SU(2)$ is generated by the operators $\hat{s}_i = \frac{1}{2}\Sigma^{(i)}$ satisfying the commutation rules (88) and $[\mathcal{J}_i,\,\hat{s}_j] = i\varepsilon_{ijk}\hat{s}_k$. They generates the $SU(2)$ matrices (89),

$$T(\rho) = e^{-i\vec{\rho}\cdot\vec{\Sigma}} = \left(\begin{array}{cc} \hat{U}(\vec{\rho}) & 0 \\ 0 & \mathbf{1}_2 \end{array}\right), \tag{191}$$

where $\vec{\rho} = \alpha\vec{\nu}$ ($\vec{\nu}^2 = 1$) and $\hat{U}$ is the $SU(2)$ transformation

$$\hat{U}(\vec{\rho}) = e^{-i\vec{\rho}\cdot\vec{\sigma}} = \mathbf{1}_2\cos\alpha - i\vec{\nu}\cdot\vec{\sigma}\sin\alpha\,. \tag{192}$$

These matrices give the transformations (92) and (93). Furthermore, we have to construct the group $Aut(\boldsymbol{d}_{\boldsymbol{f}}) = vect(G_{\boldsymbol{f}}) \circledS O_{\boldsymbol{f}}$ where $O_{\boldsymbol{f}}$ is the group of the real-valued orthogonal matrices defined by Eq. (98). Using the results of Theorem 23, it is straightforward to calculate these matrices in the gauge (140) obtaining $\hat{\mathfrak{R}}(\xi) = \mathfrak{R}^T$ for any isometry $\phi_\xi = \phi_{\mathfrak{R},a^4}$. This explains why D^{i} behave under rotations as vector components while D remains invariant. In other respects, from Eq. (191) we find that the discrete transformations of the group $\mathbb{Q}(\boldsymbol{f})$ are $\mathbf{1}$, γ^5 and the sets of matrices $U_{(k)} = \mathrm{diag}(i\sigma_k, \mathbf{1}_2)$ and $\gamma^5 U_{(k)}$.

In current calculations, when we are not interested to exploit the $\mathcal{N} = 4$ superalgebra, it is indicated to use the simpler operators

$$Q_i = iH^{-1}D^{(i)} = H^{-1}\mathcal{Q}(\sigma_i) = \mathcal{D}(\sigma_i)\,, \tag{193}$$

instead of $D^{(i)}$. However, in this case the fourth partner of the operators Q^i is rather trivial since this is just I. Therefore, these operators form a representation of the quaternion units (or of the algebra of Pauli matrices) with values in $\boldsymbol{D}_0$,

$$Q_i\, Q_j = \delta_{ij} I + i\varepsilon_{ijk} Q_k \,, \tag{194}$$

producing an evident $\mathcal{N} = 3$ superalgebra.

The corresponding Dirac-type operator of the last K-Y tensor, f^Y, calculated according to the general rule (66) with a suitable phase factor (i), was obtained in [15]. This has the form

$$D^Y = -\mathcal{Q}(\sigma_r) + \frac{2i}{\mu\sqrt{V}} \left(\begin{array}{cc} 0 & \lambda \\ -\lambda & 0 \end{array} \right) , \tag{195}$$

where the Pauli operators $\sigma_r = \vec{\sigma}\cdot\vec{x}/r$ and $\lambda = \vec{\sigma}\cdot(\vec{x}\times\vec{P}) + \mathbf{1}_2 = \sigma_L + \mathbf{1}_2 + \mu\sigma_r P_4$ have the properties

$$\{\sigma_r, \lambda\} = 0 \,, \quad [\sigma_r, \sigma_P] = \frac{2i}{r}\, \lambda \,, \tag{196}$$

$$\sigma_P \lambda = -\lambda\sigma_P = \frac{i}{2}\, \vec{\sigma}\cdot(\vec{P}\times\vec{L} - \vec{L}\times\vec{P}) - \frac{i\mu}{r}\lambda P_4 \,, \tag{197}$$

that help one to find the equivalent forms reported in [15] and verify that D^Y commutes with H and P_4 and anticommutes with D and γ^5. Moreover, after a little calculation, we obtain the remarkable identity

$$\mu P_4 \left[D^Y + \mathcal{Q}(\sigma_r) \right] = \left\{ H,\, \hat{\Lambda} \right\} \,, \tag{198}$$

involving the operator $\hat{\Lambda} = \mathrm{diag}(\lambda, \lambda)$ that is a particular version of an operator proposed by Biedenharn [58]. This is not conserved but $\hat{\Lambda}^2 = \vec{\mathcal{J}}^2 - \mu^2 {P_4}^2 + \frac{1}{4}\mathbf{1}$ has this property. Furthermore, we observe that, according to Eq. (197), the physical part of D^Y can be put in the form

$$D^Y I = \mathcal{Q}\left(-\sigma_r + \frac{2i}{\mu}\lambda\pi\Delta^{-1} \right) = \mathcal{Q}(\sigma^Y \Delta^{-1}) \,, \tag{199}$$

where

$$\sigma^Y = \frac{2}{\mu}\, [\sigma_K + (\sigma_L + \mathbf{1}_2) P_4] \tag{200}$$

is a new conserved Pauli operator associated to

$$Q^Y = H D^Y = H D^Y I = \mathcal{D}(\sigma^Y) \in \boldsymbol{D}_0. \tag{201}$$

We note that the Pauli operators $\sigma_L = \vec{\sigma}\cdot\vec{L}$ and $\sigma_K = \vec{\sigma}\cdot\vec{K}$ are conserved and satisfy

$$\{\sigma_K,\, \sigma_L + \mathbf{1}_2\} = 0 \,, \quad \{\sigma_r,\, \sigma_L + \mathbf{1}_2\} = -2\mu P_4 \,. \tag{202}$$

As in the case of the Klein-Gordon theory, we can define the components of the conserved Runge-Lenz operator of the Dirac theory [15, 16] giving directly their physical parts,

$$\mathcal{K}_i I = \frac{i\mu}{4}\left\{HD^Y,\, H^{-1}D^{(i)}\right\} + \frac{i}{2}(\mathcal{B}-P_4)H^{-1}D^{(i)} - \mathcal{J}_i I P_4\,, \tag{203}$$

where $\mathcal{B}^2 = {P_4}^2 - H^2$. Since $\mathcal{B}^2 I = \mathcal{D}(B^2)$, we can express

$$\mathcal{K}_i I = \mathcal{D}(\hat{K}_i)\,, \quad \hat{K}_i = K_i + \frac{\sigma_i}{2}\,B \in \boldsymbol{P}\,. \tag{204}$$

Furthermore, we obtain the following commutation relations

$$\begin{array}{rcl} [\mathcal{J}_i,\, \mathcal{J}_j] & = & i\varepsilon_{ijk}\mathcal{J}_k\,, \\ [\mathcal{J}_i,\, \mathcal{K}_j] & = & i\varepsilon_{ijk}\mathcal{K}_k\,, \\ [\mathcal{K}_i,\, \mathcal{K}_j] & = & i\varepsilon_{ijk}\mathcal{J}_k\mathcal{B}^2\,, \end{array} \tag{205}$$

and the commutators with the operators Q_i [18],

$$[\mathcal{J}_i,\, Q_j] = i\varepsilon_{ijk}Q_k\,, \qquad [\mathcal{K}_i,\, Q_j] = i\varepsilon_{ijk}Q_k\mathcal{B}\,. \tag{206}$$

Other useful operators related to $\vec{\mathcal{K}}$ are $\mathcal{C}_1 = \vec{\mathcal{J}}^2\mathcal{B}^2 + \vec{\mathcal{K}}^2$, $\mathcal{C}_2 = \vec{\mathcal{J}}\cdot\vec{\mathcal{K}}$ as well as

$$\mathcal{Q} = 2\vec{\mathcal{J}}\cdot\vec{\mathcal{K}} - \tfrac{1}{2}\mathcal{B}\,, \tag{207}$$

the physical part of which is

$$\mathcal{Q}I = \frac{\mu}{2}HQ^Y + (\mathcal{B}-P_4)\mathcal{D}(\sigma_L + \mathbf{1}_2) = \mathcal{D}[\sigma_K + (\sigma_L + \mathbf{1}_2)B]\,. \tag{208}$$

These operators represent Casimir-type operators for the open algebra (205), the last one obeying the simple algebraic relations,

$$[\mathcal{Q}, \mathcal{J}_i] = 0\,, \quad [\mathcal{Q}, \mathcal{K}_i] = 0\,, \quad \{\mathcal{Q}, Q_i\} = 2(\mathcal{K}_i + \mathcal{J}_i\mathcal{B})I\,, \tag{209}$$

and the identities

$$\mathcal{Q}^2 = \frac{\mu^2}{4}({P_4}^2 + \mathcal{B}^2)^2\,, \quad \mathcal{C}_1 = \mathcal{Q}^2 + \mathcal{B}\mathcal{Q} - \frac{1}{2}\mathcal{B}^2\,, \tag{210}$$

resulting from Eqs. (150) and (208).

We conclude that the conserved operators of the Dirac theory can be associated with conserved Pauli or Klein-Gordon operators produced by the same geometrical objects. A brief image of these relationships is given in the table below.

geometric object	nature	symmetry	Dirac operator	Pauli operator	Klein-Gordon operator
$f^{(i)}_{\mu\nu}$	K-Y tensor	*	Q_i	σ_i	-
$f^{Y}_{\mu\nu}$	K-Y tensor	*	Q^Y	σ^Y	-
$k^{\mu}_{(4)}$	K vector	$U(1)_4$	P_4	P_4	P_4
$k^{\mu}_{(i)}$	K vector	$SO(3)$	$\mathcal{J}_i$	J_i	L_i
$k^{\mu\nu}_{(i)}$	S-K tensor	hidden	$\mathcal{K}_i$	$\hat{K}_i$	K_i

Now we may ask how could be organized this very rich set of conserved Dirac operators. There are many commutation and anticommutation relations that cannot be ignored such that it seems that the suitable structure may be a superalgebra. The main pieces here are the operators Q_i generating an usual $\mathcal{N}=3$ superalgebra and the operators $\mathcal{J}_i$ and $\mathcal{K}_i$ giving dynamical algebras under suitable rescalings. However, if one embeds only these operators, without taking into account the operator Q, one restricts to a particular superalgebra which seems to be rather trivial. On the other hand, this operator obey the Eqs. (210) that lead to complications which cannot be removed by an usual rescaling, similar to that used in the scalar theory. This suggests us that the suitable algebraic structure involving all the above ingredients may be an *infinite loop* superalgebra constructed as in [59].

Here we propose a version of a such superalgebra as an argument for further investigations of the new types of infinite algebraic structures [60]. Let us start with the definitions of the *bosonic* operators

$$I_n = \mathcal{B}^n I\,, \quad J^i_n = \mathcal{J}_i \mathcal{B}^n I\,, \quad K^i_n = \mathcal{K}_i \mathcal{B}^n I\,, \tag{211}$$

and the supercharges of the *fermionic* sector

$$Q_n = Q\mathcal{B}^n I\,, \qquad Q^i_n = Q_i \mathcal{B}^n I\,, \tag{212}$$

for all $n = 0,1,2,....$ Then, according to Eqs. (205) and (211), we obtain the following commutators of the bosonic sector

$$[I_n, I_m] = 0\,, \qquad \left[J^i_n, J^j_m\right] = i\varepsilon_{ijk} J^k_{n+m}\,, \tag{213}$$

$$\left[I_n, J^i_m\right] = 0\,, \qquad \left[J^i_n, K^j_m\right] = i\varepsilon_{ijk} K^k_{n+m}\,, \tag{214}$$

$$\left[I_n, K^i_m\right] = 0\,, \qquad \left[K^i_n, K^j_m\right] = i\varepsilon_{ijk} J^k_{n+m+2}\,, \tag{215}$$

while from Eqs. (194) and (209) and we deduce the anticommutors of the fermionic sector,

$$\{Q^i_n, Q^j_m\} = 2\delta_{ij} I_{n+m}\,, \tag{216}$$

$$\{Q_n, Q^i_m\} = 2(K^i_{n+m} + J^i_{n+m+1})\,, \tag{217}$$

$$\{Q_n, Q_m\} = c_0 I_{n+m} + c_1 I_{n+m+2} + c_2 I_{n+m+4}\,, \tag{218}$$

where

$$c_0 = \frac{\mu^2 \hat{q}^4}{2}\,, \quad c_1 = \mu^2 \hat{q}^2\,, \quad c_2 = \frac{\mu^2}{2} \tag{219}$$

are the structure constants resulted from Eq. (210) when the eigenvalue $\hat{q}$ of P_4 is fixed. The commutations relations between the bosonic and fermionic operators are

$$[Q_n, I_m] = 0\,, \qquad [Q_n^i, I_m] = 0\,, \tag{220}$$

$$\left[Q_n, J_m^j\right] = 0\,, \qquad [Q_n^i, J_m^j] = i\varepsilon_{ijk} Q_{n+m}^k\,, \tag{221}$$

$$\left[Q_n, K_m^j\right] = 0\,, \qquad [Q_n^i, K_m^j] = i\varepsilon_{ijk} Q_{n+m+1}^k\,. \tag{222}$$

From this infinite loop superalgebra one can extract the algebras or subalgebras one needs, including the dynamical algebras governing the quantum modes corresponding to the different spectral domains of H.

4.6 The quantum modes of the Dirac field

The large collection of conserved observables we dispose will help us to select many different complete sets of commuting observable which should define new types of static quantum modes. Here we restrict ourselves to discuss only the sets including the operators H and P_4 (or P_χ) for which more three operators are needed in order to completely determine the quantum modes with given energy $E > 0$ and $\hat{q}$. These operators can be selected at the level of the associated Pauli operators since, according to Theorem 24, the eigenvalue problem $H\psi_E = E\psi_E$ is solved by the spinors $\psi_E = (u_E,\, E^{-1}\boldsymbol{\alpha} u_E)^T$ where u_E satisfy $\Delta u_E = E^2 u_E$. Therefore, in order to well-define the Pauli spinors u_E we have to chose, in addition, three operators from $\boldsymbol{P}$ which will complete the set of the commuting operators allowing u_E as common eigenspinors. In this way the problem is solved since the second Pauli spinor of ψ_E is $E^{-1}\boldsymbol{\alpha} u_E$. The last step is to identify the operators from $\boldsymbol{D}$ whose physical parts form the set of commuting Dirac operators associated to the Pauli ones.

In what follows we shall define many types of quantum modes with given energy starting with the simplest ones for which u_E has separated variables.

Definition 21 *We say that the common eigenspinors of the set of commuting operators* $\{H,\, P_\chi,\, \vec{\mathcal{J}}^2,\, \mathcal{J}_3,\, \mathcal{Q}(\sigma_L + \mathbf{1}_2)\}$ *corresponding to the eigenvalues* E, q, $j(j+1)$, m_j *and* $\pm E(j + \frac{1}{2})$. *define the* simple central *modes.*

Therefore, u_E must be the common eigenspinor of the set $\{\Delta,\, P_\chi,\, \vec{J}^2,\, J_3,\, \sigma_L + \mathbf{1}_2\}$ corresponding to the eigenvalues E^2, q, $j(j+1)$, m_j and $\pm(j + \frac{1}{2})$

This problem can be solved only defining new spherical spinors involving our previously presented $SO(3) \otimes U(1)$ spherical harmonics [8]. Following the traditional method [57] we defined the spherical spinors $\Phi^{\pm}_{q,j,m_j}(\theta, \varphi, \chi)$ as the common eigenspinors of the eigenvalue problems

$$\begin{aligned}
P_\chi\,\Phi^{\pm}_{q,j,m_j} &= q\,\Phi^{\pm}_{q,j,m_j}\,, &(223)\\
\vec{J}^2\,\Phi^{\pm}_{q,j,m_j} &= j(j+1)\,\Phi^{\pm}_{q,j,m_j}\,, &(224)\\
J_3\,\Phi^{\pm}_{q,j,m_j} &= m_j\,\Phi^{\pm}_{q,j,m_j}\,, &(225)\\
(\sigma_L+\mathbf{1}_2)\,\Phi^{\pm}_{q,j,m_j} &= \pm(j+1/2)\,\Phi^{\pm}_{q,j,m_j}\,. &(226)
\end{aligned}$$

These spinors are, in addition, eigenfunctions of $\vec{L}^2$ corresponding to the eigenvalues $l(l+1)$ with $l=j\pm\frac{1}{2}$. For $j=l+\frac{1}{2}>|q|-\frac{1}{2}$ we have [57, 19]

$$\Phi^{+}_{q,j,m_j}=\frac{1}{\sqrt{2j}}\left(\begin{array}{c}\sqrt{j+m_j}\,Y^q_{j-\frac{1}{2},m_j-\frac{1}{2}}\\ \sqrt{j-m_j}\,Y^q_{j-\frac{1}{2},m_j+\frac{1}{2}}\end{array}\right), \tag{227}$$

while for $j=l-\frac{1}{2}>|q|-\frac{3}{2}$ we get

$$\Phi^{-}_{q,j,m_j}=\frac{1}{\sqrt{2j+2}}\left(\begin{array}{c}\sqrt{j-m_j+1}\,Y^q_{j+\frac{1}{2},m_j-\frac{1}{2}}\\ -\sqrt{j+m_j+1}\,Y^q_{j+\frac{1}{2},m_j+\frac{1}{2}}\end{array}\right). \tag{228}$$

These spherical spinors are orthonormal since the $SO(3)\otimes U(1)$ harmonics are orthonormal with respect to the angular scalar product (159). From the second of Eqs. (202) it results the useful property

$$\sigma_r\Phi^{\pm}_{q,j,m_j}=\pm\lambda^q_j\,\Phi^{\pm}_{q,j,m_j}+\sqrt{1-(\lambda^q_j)^2}\,\Phi^{\mp}_{q,j,m_j}\,, \tag{229}$$

where $\lambda^q_j=q/(j+\frac{1}{2})$. Note that a similar formula is reported in Ref. [61].

The energy eigenspinors of the simple central modes can be expressed in terms of these new spherical spinors as [8]

$$\begin{aligned}
\psi^{\pm}_{E,q,j,m_j} &= \left(\begin{array}{c} u^{\pm}_{E,q,j,m_j}\\ E^{-1}\boldsymbol{\alpha}u^{\pm}_{E,q,j,m_j}\end{array}\right)\\
&= N_{E,q,j}\,\frac{1}{r}\left[\begin{array}{c}\mathfrak{f}^{\pm}_{E,q,j}\,\Phi^{\pm}_{q,j,m_j}\\ iE^{-1}\sqrt{V}\left(\mathfrak{h}^{\pm}_{E,q,j}\,\Phi^{\pm}_{q,j,m_j}+\mathfrak{g}^{\pm}_{E,q,j}\,\Phi^{\mp}_{q,j,m_j}\right)\end{array}\right], \qquad (230)
\end{aligned}$$

where the radial functions $\mathfrak{f}^{\pm}_{E,q,j}\equiv\mathfrak{f}_{E,q,l_\pm}$ are the solutions of the radial equation (166) for $l=l_\pm=j\mp\frac{1}{2}$. The other radial functions have to be calculated using Eq. (229) and taking into account that

$$i\sigma_P=\sigma_r\left(\partial_r+\frac{1}{r}-\frac{\sigma_L+1}{r}\right)+\frac{1}{r}P_\chi\,. \tag{231}$$

In this way we obtain the equations

$$\mathfrak{g}^{\pm}_{E,q,j} = \sqrt{1-(\lambda^q_j)^2}\left(-\frac{d}{dr} \pm \frac{j+\frac{1}{2}}{r}\right)\mathfrak{f}^{\pm}_{E,q,j}\,, \tag{232}$$

$$\mathfrak{h}^{\pm}_{E,q,j} = \lambda^q_j\left(\mp\frac{d}{dr} + \frac{j+\frac{1}{2}}{\mu V}\right)\mathfrak{f}^{\pm}_{E,q,j}\,, \tag{233}$$

which lead to the identity

$$\mathfrak{h}^{\pm}_{E,q,j} = \frac{q}{\mu}\,\mathfrak{f}^{\pm}_{E,q,j} \pm \frac{\lambda^q_j}{\sqrt{1-(\lambda^q_j)^2}}\,\mathfrak{g}^{\pm}_{E,q,j}\,. \tag{234}$$

The simple axial modes can be constructed in a similar manner.

Definition 22 *The common eigenspinors of the set* $\{H,\ P_4,\ \mathcal{K}_3,\ \mathcal{J}_3,\ \mathcal{Q}(\sigma_3)\}$ *corresponding to the eigenvalues* E, $\hat{q}$, $\hat{\kappa}$, m_j *and* $E\sigma$ *define the* simple axial *modes.*

Now u_E is the common eigenspinor of the set $\{\Delta,\ P_4,\ \hat{K}_3,\ J_3, \sigma_3\}$ corresponding to the eigenvalues E^2, $\hat{q}$, $\hat{\kappa}$ and m_j and σ. These eigenspinors can be calculated in parabolic coordinates as axial solutions of the Klein-Gordon (or Schrödinger equation [19]) multiplied with the constant eigenspinors of $\frac{1}{2}\sigma_3$ having the eigenvalues $\sigma = \pm\frac{1}{2}$. For example, solving the problem of the discrete axial modes [19] one finds that the scalar parts are eigenfunctions of the set $\{\Delta, P_4, K_3, L_3\}$ corresponding to the eigenvalues E^2, $\hat{q}$, κ and m that give those of the Dirac modes as $\hat{\kappa} = \kappa + |b|\sigma$ and $m_j = m + \sigma$.

In other respects, the algebraic method based on the dynamical algebras will lead to new results even if the rescaling of the Runge-Lenz operator has to be done as in the scalar case for the same spectral domains of the Kepler-type problem. If we define

$$\mathcal{R}_i = \left\{\begin{array}{lllll} \mathcal{B}^{-1}\mathcal{K}_i & \text{for} & \mu<0 & \text{and} & E<|\hat{q}| \\ \mathcal{K}_i & \text{for} & \text{any } \mu & \text{and} & E=|\hat{q}| \\ i\mathcal{B}^{-1}\mathcal{K}_i & \text{for} & \text{any } \mu & \text{and} & E>|\hat{q}| \end{array}\right. , \tag{235}$$

then the operators $\mathcal{J}_i$ and $\mathcal{R}_i$ ($i = 1, 2, 3$) will generate either a representation of the $o(4)$ algebra for the discrete energy spectrum in the domain $E < |\hat{q}|$ or a representation of the $o(3,1)$ algebra for continuous spectrum in the domain $E > |\hat{q}|$. The dynamical algebra $e(3)$ corresponds only to the ground energy of the continuous spectrum, $E = |\hat{q}|$.

In order to illustrate how works the mechanism of the dynamical algebra at the level of the Dirac theory we focus on the representations of the $o(4)$ dynamical algebra of the discrete quantum modes. In the Dirac theory the dynamical algebra is the same as in the scalar case but its representations are generated by the operators $\mathcal{J}_i$ and $\mathcal{R}_i = \mathcal{B}^{-1}\mathcal{K}_i$ which have spin terms. According to Eq. (184), these representations are *equivalent* with those generated by the associated Pauli operators J_i and

$R_i = B^{-1}\hat{K}_i$ acting upon the first Pauli spinor. However, since $J_i = L_i + \frac{1}{2}\sigma_i$ and $R_i = K_i^{re} + \frac{1}{2}\sigma_i$, we draw the conclusion that the Dirac discrete modes are governed by the *reducible* representation

$$\left(\tfrac{n-1}{2}, \tfrac{n-1}{2}\right) \otimes \left(\tfrac{1}{2}, 0\right) = \left(\tfrac{n}{2}, \tfrac{n-1}{2}\right) \oplus \left(\tfrac{n}{2} - 1, \tfrac{n-1}{2}\right) . \tag{236}$$

The Casimir operators, $\hat{C}_1 = \vec{J}^2 + \vec{R}^2$ and $\hat{C}_2 = \vec{J} \cdot \vec{R}$ take now the eigenvalues, $\hat{c}_1$ and $\hat{c}_2$, such that $\hat{c}_1 - 2\hat{c}_2 = n^2 - 1$ for both irreducible representations while $\hat{c}_2 = \frac{1}{4}(2n+1)$ for the representation $(\frac{n}{2}, \frac{n-1}{2})$ and $\hat{c}_2 = -\frac{1}{4}(2n-1)$ for the representation $(\frac{n}{2} - 1, \frac{n-1}{2})$. This suggests us to use the new Pauli operator

$$\hat{C} = 2\hat{C}_2 - \frac{1}{2}\mathbf{1}_2 = \sigma_R + \sigma_L + \mathbf{1}_2 \,, \quad \sigma_R = \vec{\sigma} \cdot \vec{R} \,, \tag{237}$$

in order to distinguish between the irreducible representations resulted from the decomposition (236). The advantage is that this operator has the simplest eigenvalues, $c = \pm n$.

The new Dirac operator $\mathcal{C} = 2\vec{\mathcal{J}} \cdot \vec{\mathcal{R}} - \frac{1}{2}\mathbf{1}$, associated to $\hat{C}$, allows us to define new quantum modes.

Definition 23 *We say that the set* $\{H,\, P_4,\, \mathcal{C},\, \vec{\mathcal{J}}^2,\, \mathcal{J}_3\}$ *defines the* natural central *modes whose eigenspinors correspond to the eigenvalues* E_n, $\hat{q}$, c, $j(j+1)$ *and* m_j. *Another possibility is to take the set* $\{H,\, P_4,\, \mathcal{C},\, \mathcal{R}_3,\, \mathcal{J}_3\}$ *of the* natural axial *modes corresponding to the eigenvalues* E_n, $\hat{q}$, c, m_r *and* m_j.

Since neither $\mathcal{Q}(\sigma_L + \mathbf{1}_2)$ nor Q_3 do not commute with $\mathcal{C}$, it results that the natural modes do not have eigenspinors with separated variables. Therefore these must be linear combinations of the spinors of simple modes. Then it is interesting to try to write down these linear combinations using only algebraic methods.

We observe the first Pauli spinors, $u_{n,\hat{q},c,j,m_j}$, of the eigenspinors $\psi_{n,\hat{q},c,j,m_j}$ of the natural central modes must be the eigenspinors of the set $\{\Delta,\, P_4,\, \hat{C},\, \vec{J}^2,\, J_3\}$ corresponding to the eigenvalues E_n^2, $\hat{q}$, c, $j(j+1)$ and m_j. On the other hand, the superalgebra

$$\{\sigma_R,\, \sigma_L + \mathbf{1}_2\} = 0 \,, \quad (\sigma_R)^2 = \hat{C}_1 - \vec{L}^2 - \sigma_L \,, \tag{238}$$

allows us to demonstrate that the first Pauli spinors of the eigenspinors of the simple central modes (230) have the remarkable property

$$\sigma_R\, u^{\pm}_{n,\hat{q},j,m_j} = \left[n^2 - \left(j + \tfrac{1}{2}\right)^2\right]^{1/2} u^{\mp}_{n,\hat{q},j,m_j} \,. \tag{239}$$

Hereby it results that the Dirac eigenspinors of the natural central modes can be expressed as

$$\begin{aligned} \psi_{n,\hat{q},c=\pm n,j,m_j} \quad = \quad & \frac{1}{\sqrt{2n}} \left[\pm\sqrt{n \pm \left(j + \tfrac{1}{2}\right)}\, \psi^{\pm}_{n,\hat{q},j,m_j} \right. \\ & \left. + \sqrt{n \mp \left(j + \tfrac{1}{2}\right)}\, \psi^{\mp}_{n,\hat{q},j,m_j} \right] . \end{aligned} \tag{240}$$

In the same way the eigenspinors of the natural axial modes can be expressed as linear combinations of the spinors of the simple axial modes. As in previous case the calculations reduce to the eigenspinors of the set of Pauli operators $\{\Delta,\, P_4,\, \hat{C},\, R_3,\, J_3\}$ associated to that of the Dirac operators of the natural axial modes. Using the identity $\{\hat{C},\, \sigma_3\} = 2(R_3 + J_3)$ it is not difficult to show that the eigenspinors of the natural axial modes are

$$\begin{aligned}\psi_{n,\hat{q},c=\pm n,m_r,m_j} &= \frac{1}{\sqrt{2n}}\left[\pm\sqrt{n \pm |m_r + m_j|}\,\psi^{\pm}_{n,\hat{q},m_r,m_j}\right. \\ &\qquad \left. +\sqrt{n \mp |m_r + m_j|}\,\psi^{\mp}_{n,\hat{q},m_r,m_j}\right], \qquad (241)\end{aligned}$$

where $m_r = |b|^{-1}\hat{\kappa}$.

Thus we see that in the Dirac theory the algebraic method offers us the mechanisms of constructing new quantum modes having no separated variables. We can say that this method and that of separation of variables complete to each other, helping us to find many types of different quantum modes related among themselves.

Appendix A
Kählerian geometries

Let us consider the manifold M_n ($n = 2k$) and its tangent fiber bundle, $\mathcal{T}(M_n)$, assuming that M_n is equipped with a *complex structure* that is a particular bundle automorphism $h : \mathcal{T}(M_n) \to \mathcal{T}(M_n)$ which satisfies $\langle h\rangle^2 = -\boldsymbol{I}$ and is covariantly constant. Notice that the matrix of h in local frames is an orthogonal point-dependent transformation of the gauge group $G(\eta)$. With its help one gives the following definition [62, 45]:

Definition 24 *A Riemannian metric g on M_n is said Kählerian if h is pointwise orthogonal, i.e., $g(hX, hY) = g(X, Y)$ for all $X, Y \in \mathcal{T}_x(M_n)$ at all points x.*

In local coordinates, h is a skew-symmetric second rank tensor with real-valued components, $h_{\mu\nu} = -h_{\nu\mu}$, which obey $g_{\mu\nu}h^{\mu\,\cdot}_{\cdot\,\alpha}h^{\nu\,\cdot}_{\cdot\,\beta} = g_{\alpha\beta}$. This gives rise to the symplectic form $\tilde{\omega} = \frac{1}{2}h_{\nu\mu}dx^{\nu} \wedge dx^{\mu}$ (i.e., closed and non-degenerate). Alternative definitions can be formulated starting with both, g and $\tilde{\omega}$, which have to satisfy the Kähler relation $\tilde{\omega}(X, Y) = g(X, hY)$ [45].

A *hypercomplex structure* on M_n is an ordered triplet $H = (h^1, h^2, h^3)$ of complex structures on M_n satisfying Eq. (87). In Lie algebraic terms, the matrices $\frac{1}{2}\langle h^j\rangle$ realize the $su(2)$ algebra.

Definition 25 *A hyper-Kähler manifold is a manifold whose Riemannian metric is Kählerian with respect to each different complex structures h^1, h^2 and h^3.*

Our unit roots, f, are defined in a similar way as the complex structures with the difference that the unit roots are automorphisms of the *complexified* tangent bundle, $f : \mathcal{T}(M_n)\otimes\mathbb{C} \to \mathcal{T}(M_n)\otimes\mathbb{C}$. Therefore, f have complex-valued components and the transformation matrix $\langle f\rangle$ is of the complexified group $G_c(\eta)$. Thus it is clear that the real-valued unit roots are complex structures as defined above. The families of unit roots may differ from the hypercomplex structures but have the same algebraic properties given by Eq. (87).

The passing from the complex structures to unit roots is productive from the point of view of the Dirac theory since in this way one can introduce families of unit roots generating superalgebras of Dirac-type operators even in manifolds which do not admit complex structures. The Minkowski spacetime is a typical example.

Appendix B
The Minkowski spacetime

In the Minkowski spacetime, M_{3+1}, with the metric $\eta = (1, -1, -1, -1)$ we consider the gauge of the *inertial* frames, $e^\mu_\nu = \hat{e}^\mu_\nu = \delta^\mu_\nu$, $(\mu, \nu, ... = 0, 1, 2, 3)$. In this gauge we use the chiral representation of the Dirac matrices (with off-diagonal $\gamma = \gamma^0$ [57]) where the standard Dirac operator reads

$$D = i\gamma^\mu\partial_\mu = \left(\begin{array}{cc} 0 & i(\partial_t + \vec{\sigma}\cdot\vec{\partial}) \\ i(\partial_t - \vec{\sigma}\cdot\vec{\partial}) & 0 \end{array} \right) = \left(\begin{array}{cc} 0 & D^{(+)} \\ D^{(-)} & 0 \end{array} \right) , \tag{B.1}$$

and the generators of the spinor representation of the group $\boldsymbol{G}(\eta) = SL(2, \mathbb{C})$ take the form $S^{ij} = \varepsilon_{ijk}S_k = \frac{1}{2}\varepsilon_{ijk}\mathrm{diag}(\sigma_k, \sigma_k)$ and $S^{i0} = \frac{i}{2}\mathrm{diag}(\sigma_i, -\sigma_i)$.

The isometries of M_{3+1} are just the transformations $x' = \Lambda(\omega)x - a$ of the Poincaré group, $\mathcal{P}^\uparrow_+ = T(4) \circledS L^\uparrow_+$ [21]. If we denote by $\xi^{(\mu\nu)} = \omega^{\mu\nu}$ the $SL(2, \mathbb{C})$ parameters and by $\xi^{(\mu)} = a^\mu$ those of the translation group $T(4)$, then we obtain the standard basis generators

$$\begin{aligned} X^\rho_{(\mu)} &= i\partial_\mu\,, && \text{(B.2)}\\ X^\rho_{(\mu\nu)} &= i(\eta_{\mu\alpha}x^\alpha\partial_\nu - \eta_{\nu\alpha}x^\alpha\partial_\mu) + S_{\mu\nu}\,, && \text{(B.3)} \end{aligned}$$

which show us that in this gauge ψ transforms manifestly covariant. On the other hand, it is clear that the group $S(M_{3+1}) = \tilde{\mathcal{P}}^\uparrow_+ \sim T(4) \circledS SL(2, \mathbb{C})$ is just the universal covering group of $I(M_{3+1}) = \mathcal{P}^\uparrow_+$. In applications it is convenient to denote $\mathcal{J}_i = \frac{1}{2}\varepsilon_{ijk}X_{jk}$ and $\mathcal{K}_i = X_{0i}$.

The Minkowski spacetime possesses a pair of adjoint triplets [10]. The unit roots of the first triplet, $\boldsymbol{f}$, have the non-vanishing complex-valued components [10]

$$f^{(1)}_{23} = 1\,, \qquad f^{(2)}_{31} = 1\,, \qquad f^{(3)}_{12} = 1\,, \tag{B.4}$$

$$f^{(1)}_{01} = i\,, \qquad f^{(2)}_{02} = i\,, \qquad f^{(3)}_{03} = i\,, \tag{B.5}$$

giving rise to the spin-like operators

$$\Sigma^{(i)} = \frac{1}{2} f^{(i)}_{\mu\nu} S^{\mu\nu} = \begin{pmatrix} \sigma_i & 0 \\ 0 & 0 \end{pmatrix}, \tag{B.6}$$

and to the Dirac-type operators

$$D^{(i)} = i[D, \Sigma^{(i)}] = \begin{pmatrix} 0 & -i\sigma_i D^{(+)} \\ iD^{(-)}\sigma_i & 0 \end{pmatrix}, \tag{B.7}$$

which *anticommute* with each other as well as with D and γ^0.

The operators D and D^i form a basis for the superalgebra $(\boldsymbol{d}_{\boldsymbol{f}})_c$ defined now over $\mathbb{C}$ since the isometries will give rise to complex-valued orthogonal transformations among $D^{(i)}$. The spinor representation of group $SL(2,\mathbb{C}) \subset S(M_{3+1})$ is generated by $\mathcal{J}_i$ and $\mathcal{K}_i$ while that of the group $(G_{\boldsymbol{f}})_c \sim SL(2,\mathbb{C})$ is generated by the operators $\hat{s}_i = \frac{1}{2}\Sigma^{(i)}$ and $\hat{r}_i = -\frac{i}{2}\Sigma^{(i)}$. All these generators satisfy usual $sl(2,\mathbb{C})$ commutation rules and

$$\begin{aligned} [\mathcal{J}_i,\, \hat{s}_j] &= i\varepsilon_{ijk}\hat{s}_k\,, \qquad [\mathcal{K}_i,\, \hat{s}_j] = i\varepsilon_{ijk}\hat{r}_k\,, \\ [\mathcal{J}_i,\, \hat{r}_j] &= i\varepsilon_{ijk}\hat{r}_k\,, \qquad [\mathcal{K}_i,\, \hat{r}_j] = -i\varepsilon_{ijk}\hat{s}_k\,. \end{aligned} \tag{B.8}$$

The next step is to construct the group $Aut(\boldsymbol{d}_{\boldsymbol{f}})_c = vect(G_{\boldsymbol{f}})_c \circledS O_{\boldsymbol{f}}$ where $O_{\boldsymbol{f}}$ is the group of the complex-valued orthogonal matrices defined by Eq. (98) that read

$$\hat{\mathfrak{R}}_{ij}(\omega) = \tfrac{1}{4} f^{(i)\,\alpha\beta} \Lambda^{\mu\,\cdot}_{\cdot\,\alpha}(\omega) \Lambda^{\nu\,\cdot}_{\cdot\,\beta}(\omega) f^{(j)}_{\mu\nu}\,. \tag{B.9}$$

These form a representation of the group $I(M_{3+1}) = \mathcal{P}^{\uparrow}_{+}$ *induced* by the group $O(3)_c$. Of course, the translations have no effects in this representation remaining only with the transformations $\Lambda(\omega) \in O(3,1)$. These give rise to non-trivial matrices $\hat{\mathfrak{R}}(\omega)$ as, for example,

$$\hat{\mathfrak{R}}(\varphi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & \sin\varphi \\ 0 & -\sin\varphi & \cos\varphi \end{pmatrix}, \quad \hat{\mathfrak{R}}(\alpha) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh\alpha & i\sinh\alpha \\ 0 & -i\sinh\alpha & \cosh\alpha \end{pmatrix}, \tag{B.10}$$

calculated for non-vanishing parameters $\omega_{23} = \varphi$ (a rotation around x^1) and respectively $\omega_{01} = \alpha$ (a boost along x^1). Hereby we see that $O_{\boldsymbol{f}} \sim O(3)_c$ which requires linear structures defined over $\mathbb{C}$ instead of $\mathbb{R}$ as we used in the case of the hyper-Kähler manifolds. Thus we have all the ingredients we need to write down the action of the transformations of the group $Aut(\boldsymbol{d}_{\boldsymbol{f}})_c$. We note that Eqs. (B.8) show that $(G_{\boldsymbol{f}})_c$ is an invariant subgroup.

The second triplet is $\boldsymbol{f}^*$ for which all the spinor quantities are just the Dirac conjugated of those of $\boldsymbol{f}$. The corresponding spin-like operators are

$$\overline{\Sigma^{(i)}} = \frac{1}{2}\left(f^{(i)}_{\mu\nu}\right)^* S^{\mu\nu} = \begin{pmatrix} 0 & 0 \\ 0 & \sigma_i \end{pmatrix}, \tag{B.11}$$

which means that the representations $spin(G_{\boldsymbol{f}})$ and $spin(G_{\boldsymbol{f}^*})$ of $SU(2)$ act *separately* on the left and right-handed parts of the Dirac spinor. Moreover, it is interesting to observe that $\Sigma^{(i)} + \overline{\Sigma^{(i)}} = 2S^i$. This perfect balance between the chiral sectors is due to the fact that the operator $D^{(+)}D^{(-)} = D^{(-)}D^{(+)}$ commutes with σ_i.

The discrete symmetry is given by two representations of the quaternion group acting on each of both chiral sectors. On the left-handed sector acts the group $\mathbb{Q}(\boldsymbol{f})$ represented by the operators $\mathbf{1}$, $F = \gamma^5 = \mathrm{diag}(-\mathbf{1}_2, \mathbf{1}_2)$, $U_{(i)} = \mathrm{diag}(i\sigma_i, \mathbf{1}_2)$ and $\gamma^5 U_{(i)}$. The operators of the group of the right-handed sector, $\mathbb{Q}(\boldsymbol{f}^*)$, can be obtained using the Dirac adjoint and taking into account that here $\overline{\gamma^5} = -\gamma^5$. The resulted operators, $\mathrm{diag}(\mathbf{1}_2, \pm i\sigma_i)$, act only on the right-handed sector.

Hence it is clear that each chiral sector has its own sets of unit roots defining Dirac-type operators. These are the spheres $S^2_{\boldsymbol{f}}$ of the left-handed sector and $S^2_{\boldsymbol{f}^*}$ of the right-handed one. Since there are no other independent unit roots, the whole set of unit roots of the Minkowski spacetime is $\boldsymbol{R}_1(M_{3+1}) = S^2_{\boldsymbol{f}} \bigcup S^2_{\boldsymbol{f}^*}$. The continuous and discrete symmetry groups of the Dirac-type operators are defined separately on each of these two spheres.

References

[1] B. Carter and R. G. McLenaghan, *Phys. Rev.* D19 (1979) 1093.

[2] D. G. Boulware *Phys. Rev.* **D12** (1975) 350.

[3] I. I. Cotăescu, *J. Phys. A: Math. Gen.* **33** (2000) 9177.

[4] I. I. Cotăescu, *Phys. Rev.* **D65** (2002) 084008.

[5] K. Yano, *Ann. Math.* **55** (1952) 328.

[6] M. Cariglia, *Class. Quantum Grav.* **21** (2004) 1051.

[7] R. Penrose, *Ann. NY Acad. Sci.* **224** (1973) 125; R. Floyd, *The dynamics of Kerr fields*, (Ph D. Thesis, London, 1973).

[8] I. I. Cotăescu and M. Visinescu, *Int. J. Mod. Phys.* **A16** (2001) 1743.

[9] V. V. Klishevich, *Class. Quantum Grav.* **17** (2000) 305.

[10] V. V. Klishevich, *Class. Quantum Grav.* **19** (2002) 4287.

[11] I. I. Cotăescu and M. Visinescu, *Class. Quantum Grav.* **43** (2004) 2987.

[12] I. I. Cotăescu and M. Visinescu, *Gen. Relat. Grav.* **35** (2003) 389.

[13] D. Vaman and M. Visinescu, *Phys. Rev.* **D57** (1998) 3790.

[14] D. Vaman and M. Visinescu, *Fortschr. Phys* **47** (1999) 493.

[15] I. I. Cotăescu and M. Visinescu, *Phys. Lett.* **B502** (2001) 229.

[16] I. I. Cotăescu and M. Visinescu, *Class. Quantum Grav.* **18** (2001) 3383.

[17] K. Peeters and A. Waldron, *J. H. E. P.* **024** (1999) 9902; `hep-th/9901016`.

[18] I. I. Cotăescu and M. Visinescu, *J. Math. Phys.* **43** (2002) 2987.

[19] I. I. Cotăescu and M. Visinescu, *Mod. Phys. Lett.* **A15** (2000) 145.

[20] I. I. Cotăescu and M. Visinescu, *Mod. Phys. Lett.* **A43** (2004) 2987.

[21] W.-K. Tung, *Group Theory in Physics* (World Sci., Philadelphia, Singapore, 1985).

[22] S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity* (Wiley, New York, 1972).

[23] R. M. Wald, *General Relativity*, (The Univ. of Chicago Press, Chicago and London, 1984).

[24] B. O'Neill, *Semi-Riemannian Geometry* (Academic Press, 1983).

[25] N. Salingaros, *J. Math. Phys* **23** (1982) 1.

[26] A. Van Proeyen, *Annals of the University of Craiova, Physics AUC* **9** (1999) 1; `hep-th/9910030` .

[27] N. D. Birrel and P. C. W. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, 1982).

[28] M. Hamermesh, *Group theory and its applications to physical problems* (Addison-Wesley, Reading MA, 1962).

[29] A. O. Barut and R. Raçzka, *Theory of Group Representations and Applications* (PWN, Warszawa 1977).

[30] G. Mackey, *Induced Representations of Groups and Quantum Mechanics* (Benjamin, New York, 1968).

[31] P. R. Carter, *Phys. Rev.* **D16** (1977) 3395.

[32] G. W. Gibbons, R. H. Rietdijk and J. W. van Holten, *Nucl. Phys.* **B404** (1993) 42.

[33] R. G. McLenaghan and Ph. Spindel, *Phys. Rev.* **D20** (1979) 409.

[34] N. Kamran and R. G. McLenaghan, *Phys. Rev.* **D30** (1984) 357.

[35] G. Gaeta and P. Morando, *J. Phys. A: Math. Gen.* **35** (2002) 3925; `math-ph/0204019`.

[36] G. W. Gibbons and P. J. Ruback, *Commun. Math. Phys.* **115** (1988) 267.

[37] M. Atiyah and N. Hitchin, *Phys. Lett.* **A107** (1985) 21; id. *The geometry and dynamics of magnetic monopoles*, (Princeton Univ. Press, 1988).

[38] T. Eguchi, P. Gilkey and A. Hanson, *Phys. Rep.* **66** (1980) 213.

[39] I. Bakas and K. Sfetsos, *Nucl. Phys. Proc. Suppl.* **49** (1996) 10; `hep-th/9601087`.

[40] I. I. Cotăescu and M. Visinescu, in preparation.

[41] I. I. Cotăescu, *Int. J. Mod. Phys.* **A19** (2004) in press.

[42] D. J. Gross and M. J. Perry, *Nucl. Phys.* **B226** (1983) 29; R. D. Sorkin, *Phys. Rev. Lett.* **51** (1983) 87; N. S. Manton,*Phys. Lett.* **B110** (1985) 54; id. *Phys. Lett.* **B154** (1985) 397.

[43] S. W. Hawking, *Phys. Lett.* **A60** (1977) 81.

[44] G. W. Gibbons and C. A. R. Herdeiro, *Class. Quant. Grav.* **16** (1999) 3619.

[45] G. W. Gibbons and N. S. Manton, *Nucl. Phys.* **B274** (1986) 183.

[46] G. W. Gibbons and P. J. Ruback, *Phys. Lett.* **B188** (1987) 226.

[47] L. Gy. Feher and P. A. Horváthy, *Phys. Lett.* **B182** (1987) 183; id. (E) **B188** (1987) 512.

[48] B. Cordani, L. Gy. Feher and P. A. Horváthy, *Phys. Lett.* **B201** (1988) 481.

[49] J. W. van Holten, *Phys. Lett.* **B342** (1995) 47.

[50] Z. F. Ezawa and A. Iwazaki, *Phys. Lett.* **138B** (1984) 81; M. Kobayashi and A. Sugamoto, *Progr. Theor. Phys.* **72** (1984) 122; A. Bais and P. Batenberg, *Nucl. Phys.* **B245** (1984) 469; A. Comtet and P. A. Horváthy, *Phys. Lett.* **B349** (1995) 49.

[51] H. Boutaleb - Joutei and A. Chakrabarti, *Phys. Rev.* **D21** (1979) 2280.

[52] E. Newman and R. Penrose, *J. Math. Phys.* **7** (1966) 863.

[53] T. T. Wu and C. N. Yang, *Nucl. Phys.* **B107** (1976) 365.

[54] K. Gosh, *Phys. Rev.* **D67**, 124027 (2003).

[55] L. G. Mardoyan, *J. Math. Phys.* **44** (2003) 4981; `quant-ph/0306168`.

[56] E. D'Hoker and L. Vinet, *Phys. Lett.* **B137** (1984) 72; F. Bloore and P. A. Horváthy, *J. Math. Phys.* **33** (1992) 1869; F. De Jonge, A. J. Macfarlane, K. Peters, and J.- W. wan Holten, *Phys. Lett.* **B359** (1995) 114.

[57] B. Thaller, *The Dirac Equation* (Springer Verlag, Berlin Heidelberg 1992).

[58] L. C. Biedenharn, *Phys. Rev.* **162** (1962) 845; L. C. Biedenharn and N. V. V. Swamy, *Phys. Rev.* **D5** (1964) 1353

[59] C. Daboul, J. Daboul and P. Slodowy, `hep-th/9408080`.

[60] L. Gy. Feher, private communication.

[61] Y. Kazama, C. N. Yang and A. S. Goldhaber, *Phys. Rev.* **D15** (1977) 2287.

[62] H. B. Lawson, Jr. and M.-L. Michelsohn, *Spin Geometry*, (Princeton Univ. Press, 1989).